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ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1935 NOVEMBER 8

Mr. J. H. Reynolds, President, in the Chair.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Clabon Walter Allen, M.Sc., Solar Physics Observatory, Cambridge (proposed by F. J. M. Stratton) ;

Francis Parry Beech, National Provincial Bank, Bethesda, Bangor (proposed by J. A. Lloyd) ;

Captain Guy Bomford, R.E., Survey of India, Dehra Dun, India (proposed by G. P. Lenox-Conyngham) ;

Percy Byerly, Professor of Seismology, University of California, Berkeley, Cal., U.S.A. (proposed by H. Jeffreys) ;

Léon Cap, Observatorio do Estado, S. Paulo, Brazil (proposed by W. M. Lindley) ;

Rev. Ronald William Cartmel, B.A., 28 Birchwood Road, Wolverhampton (proposed by S. Fellows) ;

John Edward Daly, 5 Payne Street, Barnsbury, London, N. 1 (proposed by R. Ta'Bois) ;

Dharmasiri Jayampathie Jayasinghe, B.Sc., Colombo Observatory, Ceylon (proposed by H. Jameson) ;

Alvapillai Ponnusamy Kandasamy, B.Sc., Colombo Observatory, Ceylon (proposed by H. Jameson) ;

Peter Kennedy, M.I.E.E., A.Amer.I.E.E., 77 Edmund Avenue, Unley, S. Australia (proposed by G. F. Dodwell) ;

Zdenek Kopal, 1423 U. Pernikarky, Prague (proposed by H. Slouka) ;

Miss I. Lehmann, Chief Seismologist, Geodetic Institute, Copenhagen (proposed by Harold Jeffreys) ;

Major Earl Pitt, B.A., B.Sc., B.Litt., Meadway, Whitchurch, Herefordshire (proposed by Oliver Lodge) ; and

Kiyofusa Sotome, Astronomical Observatory, Tokyo (proposed by W. M. Smart).

Two hundred and forty presents were announced as having been received since the last meeting, including :—

Cape of Good Hope, Royal Observatory, Annals, vol. XIV, pt. 1. Stellar Parallaxes (1st series), 1926–1930 ;

Catania University, Storia della Univeristà di Catania, dalle origini al giorni nostri. Quinto centenario dell'ateneo di Catania ;

Frost (Edwin Brant), An Astronomer's Life (presented by Sir Frank Dyson) ;

Greenwich, Royal Observatory, Observations of Stellar Parallax, vol. II ;

Greenwich, Royal Observatory, Second Greenwich Catalogue of Stars for 1925·0. (In two parts.) Fundamental Stars and Stars in the Zone $+32°{\cdot}0$ to $+64°{\cdot}0$;

Hamburger Sternwarte in Bergedorf, Bergedorfer Spektral-Durchmusterung der 115 nordlichen Kapteynschen Eichfelder . . . Band I. Eichfeld 1 bis 19. Deklination : $+90°$, $+75°$, $+60°$;

International Astronomical Union, Named Lunar Formations, vols. I, II, by Mary A. Blagg and K. Müller ;

Sir Isaac Newton, 1727–1927, A Bicentenary Evaluation of His Work (presented by The History of Science Society, Washington, D.C.) ;

Plaskett (J. S.), The Dimensions and Structure of the Galaxy ; being the Halley Lecture delivered on June 5, 1935 ;

Rosseland (S.), Astrophysik auf Atomtheoretischer Grundlage (presented by Professor Dingle) ;

Walters (M. H. H.), Globular Clusters (presented by Mrs. Walters).

THE VARIABLE SPECTRUM OF γ CASSIOPEIÆ.

(Fourth Paper.)

William J. S. Lockyer, M.A., Ph.D., Major (late R.A.F.).

The spectrum of γ Cassiopeiæ (H.D. 5394, 1900 R.A. $0^h\ 50^m{\cdot}7$, Dec. $+60°\ 11'$, Ptg. Mag. 2·01, Sp. Type Boemn7) has been photographed yearly at the Norman Lockyer Observatory, Sidmouth, since the year 1921 with the exception of 1922, and altogether 142 negatives have been secured.

Three papers on the variability of its spectrum have been published,* and these were confined to discussions on the behaviour of the bright components of the hydrogen lines.

* *M.N.*, **93**, 362, 1933 ; **93**, 619, 1933 ; **95**, 520, 1935.

The object of the present communication is to record the sudden appearance of a very strong and very sharp absorption line, which is the strongest and sharpest absorption line in the whole spectrum of this star.

To understand the significance of the new observation it is necessary to make a very brief résumé of the more recent appearances of the bright components of the hydrogen lines in the spectrum.

From March to November 1934 each of the hydrogen lines contained only one bright component, but after that date, on December 20, the other component appeared, the red being very much stronger than the violet component. The violet component has since then been gradually brightening up in relation to the red *in all the hydrogen lines*. This was the case in every negative taken up to July 27 of the present year. Since then the brightening up of the violet component has continued to take place in all the hydrogen lines *with the exception of* $H\zeta$, at which wave-length the violet component vanished altogether. Why $H\zeta$ should behave differently from the other hydrogen lines seemed at the time very peculiar, and there seemed no explanation to account for it.

This state of affairs continued up to October 7, when a negative taken on that date displayed the sudden appearance of a very strong and very sharp absorption line, the strongest and sharpest absorption line in the whole spectrum. This line is coincident in position with that which should be occupied by the bright violet component of $H\zeta$.

The appearance of such a strong line was so astounding that the earliest opportunity was taken to verify it by securing another photograph. This was done on October 29, and the negative displayed again this strong absorption line.

The presence of this line naturally made one again examine very carefully the negatives of the preceding months to see if its presence could be detected, as it was thought that it might have been strengthening up previous to its great intensity on October 7. A faint trace of it could be seen on the negative taken on September 10 but none on the earlier ones.

There seems little doubt that this absorption line, but most probably of much less intensity, began to come into action about July 30, and accounts for the masking of the violet component of $H\zeta$ in the negatives taken on July 30 and August 6, 10 and 20 on which the red component alone appeared.

An examination of all the Sidmouth negatives taken since 1921 has been made, and in no instance has this new absorption line been seen. The bright components of the hydrogen line at $H\zeta$ have always behaved exactly similar to those of the other hydrogen lines, as can be seen from the values of the relative intensity of the bright components published in my previous papers on this star.

This peculiar behaviour of $H\zeta$ recalls some remarks I made with regard to this line in the spectrum of ϕ Persei,* another Be-type star. There I pointed out the unusual behaviour of this line in the following words :— "A most noteworthy feature that can be observed by examining different spectra of ϕ Persei as a whole is that, while $H\zeta$ does not seem to change in

* *M.N.*, 85, 598, 1925.

intensity at all in any of the negatives covering all phases, some spectra exhibit the double bright line at $H\beta$ and the central absorption at $H\zeta$ as the *most prominent and outstanding lines*, while others display the central absorptions at $H\gamma$, $H\delta$, $H\epsilon$, $H\eta$ and $H\theta$ so conspicuously that $H\zeta$ is *no longer specially noticeable*."

From the above it is suggested that at the wave-length of $H\zeta$ (λ 3889·05) there is some extra absorption taking place which prevents the line at $H\zeta$ from behaving normally.

A similar kind of absorption occurs in other Be-type stars observed at Sidmouth, only the hydrogen line affected in this case is $H\epsilon$. The absorbing material here is Ca^+, a line of which at wave-length 3968 blends with the hydrogen line. In the case of Me-type stars Shane * has suggested that the weakening of $H\kappa$, $H\lambda$ and $H\mu$ is attributed to blends with *Fe* lines, and that of $H\xi$ to a V blend.

It will be interesting to try and find out what substance produces the strong absorption line at $H\zeta$ in this Be-type atar.

Addendum (1935 November 14)

The most probable origin of this strong absorption line at $H\zeta$ is that given by Professor A. Fowler who has suggested that it may be the neutral Helium line at λ 3888·646 (Intensity 10). This line is the strongest line of the principal series of the triplet system. The next line in this series is λ 3187·743 (Intensity 8), but this falls outside the region photographed at Sidmouth, so its presence cannot be ascertained.

As this new absorption line falls close to the position of the bright violet component of the $H\zeta$ line in γ Cassiopeiæ an attempt was made to determine the wave-length of this component.

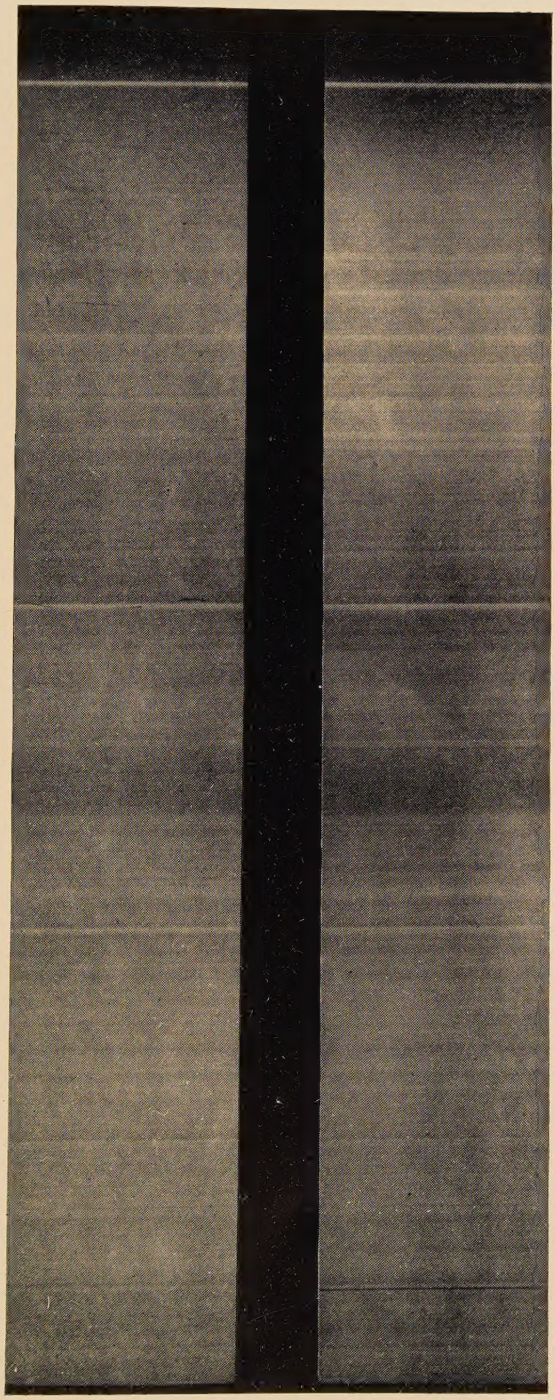
The separation of the bright components of $H\zeta$ is 3·5 A. (approx.) Assuming that the central absorption of the $H\zeta$ line (λ 3889·05) falls midway between the bright components of this line then the wave-lengths of the bright violet and red components would be approximately 3887·35 and 3890·75 respectively. Thus the neutral helium line (λ 3888·64) falls nearer the violet component and would account for the masking of this violet component.

W. J. S. L.

DESCRIPTION OF PLATE 1

The spectra, only portions of which are here shown, were taken on September 10 and October 29 of the present year, the former showing the single bright red component of the hydrogen line at $H\zeta$ with the faintest trace of the new absorption line. The latter exhibits also the single bright component and the strong absorption line at the position of the violet component. The reproductions are direct enlargements from the original negatives.

* *L.O.B.*, **10**, 131, 1921.



1. 1935 Sept. 10.
2. 1935 Oct. 29.

W. J. S. Lockyer, The Variable Spectrum of γ Cassiopeiæ.

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OBSERVATIONS OF INTENSITY WITH A SPECTROHELIOSCOPE.

R. v. d. R. Woolley and H. W. Newton.

(Communicated by the Astronomer Royal)

1. This paper describes auxiliary apparatus attached to the Hale spectrohelioscope at Greenwich by means of which it is possible to measure the light intensity of bright flocculi and prominences. The intensities of flocculi are referred to the intensity of the undisturbed solar disc at the same distance from the Sun's centre as the flocculus, and the intensities of prominences are referred to the intensity at the centre of the solar disc, in both cases in wave-lengths outside the absorption line $H\alpha$ or $H\beta$. Intensities of both flocculi and prominences can accordingly be made absolute if the solar constant and solar energy curve are known. The paper also presents the results of observations made at the Royal Observatory, Greenwich, during the period 1934 August to 1935 October. During the winter months observations of this nature are rarely possible at Greenwich.

The auxiliary apparatus is set up as follows (fig. 1). A motor-headlight filament lamp (12 v., 60 w.) is placed off the axis of the spectrohelioscope, at A in the plan. A prism C placed immediately below the main spectrohelioscope beam reflects the light from A on to the upper mirror D. This beam is dispersed by the grating E, and passes on to the lower mirror F which reflects light of the same wave-length as that in which the Sun is viewed in the spectrohelioscope. This reflected beam passes immediately above the line-shifter G and into the eyepiece H. The observer sees a small part of the spectrum of the lamp (drawn out into a band by the rotating prisms of the spectrohelioscope) immediately above the field containing that part of the solar surface which is under examination. A stop behind the first slit considerably reduces the amount of scattered light and also limits the field of view to a strip about equal in extent to that of the comparison band. An optical wedge is inserted at B, and the brightness of the lamp spectrum seen at H can be altered by adjusting this wedge. An object on the Sun—flocculus, prominence or part of the undisturbed disc—can accordingly be matched with the lamp. Observations comparing the brightness of two objects viewed in the *same* wave-length have previously been made at Greenwich simply by moving a wedge over the brighter until a match is obtained; but with the present apparatus a comparison between two objects viewed in slightly *different* wave-lengths can be made by using the lamp as an intermediary. The comparison wave-length is chosen clear of the absorption line ($H\alpha$ or $H\beta$). It is assumed that neither the brightness of the lamp nor the transparency of the sky vary during the measurement; by making observations in the order disc—flocculus—disc (the three observations being

completed within two minutes) any regular variations in lamp and sky are eliminated ; the observations must be rejected if the sky is so poor that it is not possible to get steady readings on the undisturbed disc. The lamp,

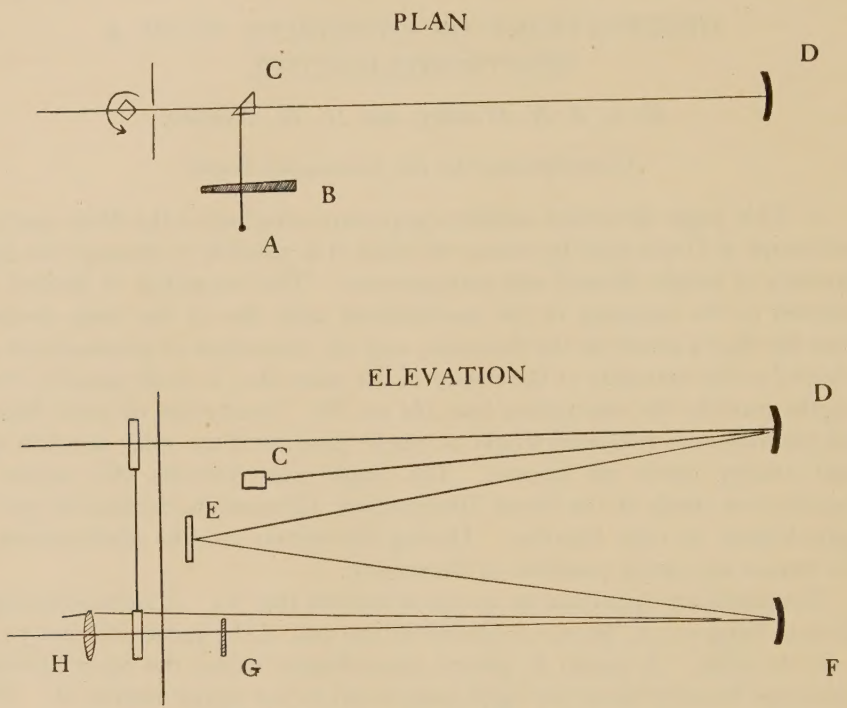

FIG. 1.—*Arrangement of Apparatus.*

which is fed by a transformer run off the mains, has given no trouble, and when the sky is moderate observations agreeing to within 7 per cent. at intervals of two minutes may be commonly obtained.

2. *Method of Measurement and Instrumental Corrections.*

(1) *Calibration of the Wedge.*—Let the wedge reading be x, the density of the wedge D, and the fraction of incident light transmitted by the wedge be f. Then the calibration curve of the wedge supplied by the makers is $D = Ax$, where A is some constant and $D = \log 1/f$. The value of the constant A (supplied by the makers, Messrs. A. Hilger, Ltd.) was checked by an application of the half-aperture method employed by Greaves, Davidson and Martin. Measures of the centre of the disc against the comparison lamp were taken (1) with the object-glass forming the solar image unobscured, and (2) and (3) with the top and bottom halves of the object-glass covered.

Let L be the brightness of the solar disc, *i.e.* the amount of light from the Sun received in the eyepiece per unit solid angle, and $B \times f$ the amount of light received from the lamp. Then in the half-aperture experiment, $L_1 = L_2 + L_3$ so that $f_1 = f_2 + f_3$. Now if the true calibration is $D = (A + \alpha)x$ it can be shown that

$$\alpha = \frac{1}{2\cdot 3}\,\frac{f_1 - f_2 - f_3}{x_1 f_1 - x_2 f_2 - x_3 f_3},$$

where x_1, x_2 and x_3 are the three settings. In practice f_1, f_2 and f_3 are taken from the old calibration of the wedge, *i.e.* $f_1 = \text{antilog}\,(-Ax_1)$, etc. A large number of observations were made on eight different dates by two observers with the object of determining the value of the correction α. The weighted mean result was $\alpha = +0\cdot004 \pm 0\cdot0014$, and the correction $\alpha = +0\cdot004$ was adopted. The makers' calibration being $D = 0\cdot149x$, the adopted calibration is accordingly $D = 0\cdot153x$.

(2) *Correction for Reflections by the Line-shifter.*—Now consider a patch on the solar disc a, having a brightness L_a, viewed in a wave-length λ_a. It is to be compared with a patch at the centre of the Sun's disc at a wave-length λ_0 (brightness L_0). Let the two settings of the wedge be x_a and x_0, and let the line-shifter be set at angles θ_a, θ_0 to the normal. The light from the comparison lamp does not pass through the line-shifter,* but the light from the Sun does so, and suffers a loss by reflection dependent on θ. Let $g(\theta)$ be the ratio of the light transmitted when the line-shifter is set at an angle θ, to the light transmitted when the line-shifter is normal.

Then

$$\frac{L_a}{L_0} = \frac{g(\theta_0)}{g(\theta_a)} \cdot \frac{f_a}{f_0} = \frac{g(\theta_0)}{g(\theta_a)}\,\frac{\text{antilog}\,(-Ax_a)}{\text{antilog}\,(-Ax_0)}.$$

The correction $g(\theta)$ used is computed from Fresnel's formula assuming a refractive index of 1·5 for the line-shifter (Table I). This formula was tested by comparing the apparent brightness of the solar disc in two wave-lengths 6576 A. and 6579 A., the former through the line-shifter in the normal position and the latter through the line-shifter inclined at an angle of 56° to the normal. The result found was $0\cdot94 \pm 0\cdot01$ in good agreement with Fresnel's formula which, for $\mu = 1\cdot5$, gives 0·93. It should be pointed out that differences in intensity on the solar energy curve over a range of 10 angstroms in the

TABLE I

Transmission through a Thin Plane Parallel Glass

Angle of Incidence	Relative Transmission
0° – 30°	100
40°	99
50°	96
52°	95
54°	94
56°	93
58°	91
60°	89

* It is necessary to make the lamp brighter than the $H\alpha$ image of the Sun. This could only be done by bringing the lamp beam through a slit wider than that of the spectrohelioscope proper.

neighbourhood of $H\alpha$ may be neglected; the present observations cannot aim at an accuracy as great as one per cent., and the change in the energy due to a change of wave-length of 10 A. is only 0·04 per cent.

3. *Measurement of the Contour of Hα*

If we confine our attention to the centre of the Sun's disc, undisturbed by any phenomenon, and make a series of comparisons with the lamp determining L_λ/L_{λ_0} distributed over various wave-lengths we shall obtain the contour of $H\alpha$. It is of interest to record observations of line contour which have been made without photographic photometry, but the spectrohelioscope is not the ideal instrument for the development of visual observations of line contour. (It is by now well known from experience with photographic methods that a monochromator should be used and that glass prisms give better results than a grating. It is intended to develop visual observations of line intensity at Greenwich using glass prisms and a monochromator.) The central intensity of $H\alpha$ found in the present series of observations is 0·23, and is undoubtedly too high, the error being due to scattered light. The general method of making the observations has already been described. In detail, since the line-shifter gives a workable range of only 4 A., a wave-length 4 A. from the centre of $H\alpha$ is taken as a secondary reference point, and every observation of intensity between this point and the centre is made between a pair of observations of the intensity of the reference point. Only one side of the contour is observed at a time, *i.e.* either the long wave-length side of $H\alpha$ or the short wave-length side is observed. The intensity of the secondary reference point is compared with the intensity at a point 10 A. from the centre of $H\alpha$—adopted as standard—in a separate experiment. Observations on the long wave-length side of $H\alpha$ were carried out on 1934 October 22, 26, 29, and 1935 September 12 and October 12: and on the short wave-length side on 1934 September 28 and October 22 and 1935 October 21 and 23. The mean contour of $H\alpha$ given by the observations is shown in Table II and fig. 2. It may be said that it agrees well with good photographic observations—those of Thackeray * and Minnaert—except that the central intensity is higher than that which Thackeray obtained with a monochromator. No correction has been applied for the distortion of the contour due to finite resolving power of the grating. It may be noted that the scale of the first order spectrum used in the present observations is 1 A. = 4·35 mm.; the grating is one ruled by Rowland and has 14,438 lines to the inch. The spectrohelioscope beam does not fill this grating, but an area containing about 43,000 lines is illuminated. The width of the slits is about 0·1 mm. The selecting agent is of course the second slit which is sampling a spectrum whose purity is determined by the grating and by the first slit. Successive points on the contour were obtained by making a series of settings with the line-shifter, the scale of which is 0–10 divisions = 0·37 A.

The curve shown in fig. 2 represents smoothed values of 186 comparisons

* *M.N.*, 95, 293, 1935.

between various wave-lengths inside the line and the secondary reference points 4 A. on either side of the centre. An individual determination may differ by as much as 10 per cent. from the mean value. The long- and short-wave halves of the contour are completely independent of each other. Depressions occur in the neighbourhood of 6560·6 A. and

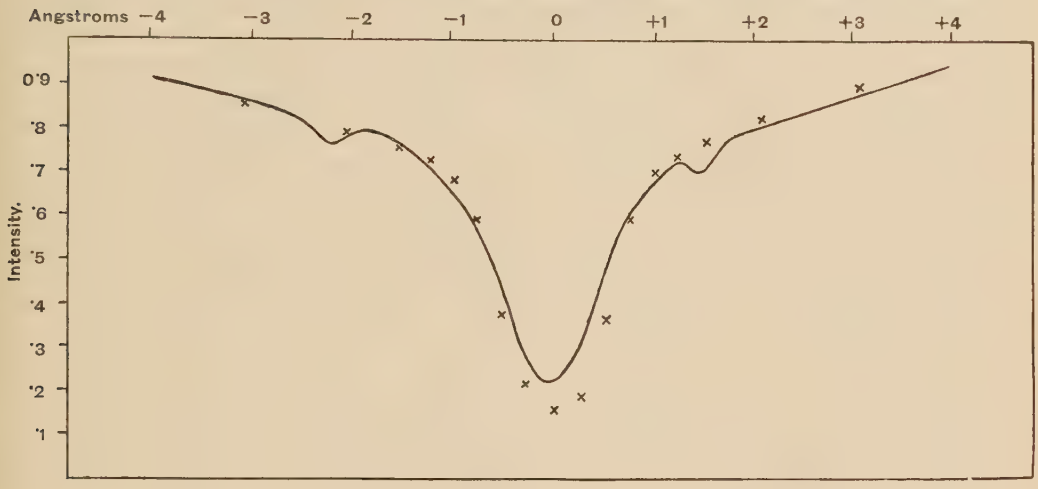

FIG. 2.—*Contour of Hα.*

Continuous curve : smoothed values of Greenwich visual observations, 1934–35.

Crosses : observed photographically by Thackeray (*M.N.*, 95, 296, 1935).

6564·2 A., where there are respectively lines of intensity 1 *Nd*?, and 0 and −1 blended, all due to atmospheric water vapour according to *Revision of Rowland's Table*. In the present observations, insufficient points were taken accurately to determine the contour of these secondary lines.

TABLE II

Contour of Hα Observed Visually

Distance from Centre of *Hα*	Intensity	
	Short Wave-length Side	Long Wave-length Side
4·0 A.	92	95
3·5	89	91
3·0	86	87
2·5	(81)	83
2·0	(78)	78
1·5	75	(71)
1·0	64	66
0·5	42	46
0·0	23	23

4. *Intensities of Flocculi*

(1) *Hα Flocculi.*—The intensities of 62 bright patches of *Hα* flocculi associated with sunspots have been compared with the intensity (=100) of

the undisturbed disc, in a wave-length 10 A. from the centre of $H\alpha$, at the same distance from the centre of the disc as each flocculus respectively. The observations are summarised in Table III. All the observed intensities of bright flocculi stand in need of some correction for scattered light. It seems probable that most scattering takes place in the form of a transfer from one wave-length to another and not from one point of the field to another. The image of the solar disc is perfectly sharp, but the light which is viewed in the eyepiece is not solely that of the wave-length indicated by the line-shifter, but contains a small percentage of light of other wave-lengths perhaps not very far removed from that of the line. Consider a continuous spectrum crossed by a single absorption line. Let I_1 be the intensity of the spectrum at some wave-length within the line and I_2 the intensity of the continuous background near the line. Let us suppose that the grating removes the fraction α of the light of any particular wave-length and scatters it uniformly in directions proper to other wave-lengths. Then the intensity given by the imperfect grating instead of I_1 is

$$I_1(1-\alpha)+\alpha I_2,$$

so that if r' is the observed intensity of a flocculus and r its true intensity (on the scale disc $=1$) we shall have for the true intensity

$$r=\frac{r'-\alpha}{1-\alpha}.$$

Taking Thackeray's value of the central intensity, $r=0{\cdot}15$ where our instrument gives $r'=0{\cdot}23$; from these values we find $\alpha=0{\cdot}09$. Returning to the mean intensity (uncorrected) of the flocculus, we have $r'=0{\cdot}36$, so that with $\alpha=0{\cdot}09$, $r=0{\cdot}30$. On account of the uncertainty of the exact nature of the scattering, we have shown all the values in this paper both uncorrected and corrected.

It is to be noticed that the correction to be applied to a prominence is quite different, as the prominence has no continuous spectrum. In this case we have simply $r=r'\,.\,\frac{1}{1-\alpha}$, and the observed brightness of prominences should be increased by about 10 per cent. In this case also, both uncorrected and corrected values will be shown.

The eruptive flocculi entered in the fourth column refer to small, very bright patches which arise suddenly within the confines of a normally bright area ; the duration of enhanced emission is of the order of a quarter to half an hour.

Table III indicates that during the period of observation the measured light intensities of normal $H\alpha$ flocculi were most frequent between 30 and 40 per cent. uncorrected (or 23 and 34 per cent. corrected), that of the continuous spectrum 10 A. away from the centre of $H\alpha$; the mean is 36 per cent. uncorrected and 30 per cent. corrected. It should be pointed out that there is probably a tendency for the observer to omit very faint material from his survey, though the faintest flocculi appear to be those in which

TABLE III

Intensities of Bright Hα Flocculi associated with Sunspots, 1934–1935

Intensity		Number of Flocculi	
Uncorrected	Corrected	Normal	Eruptive
20–24	12–16	0	
25–29	17–22	3	
30–34	23–27	21	
35–39	28–33	25	
40–44	34–38	10	1
45–49	39–44	3	3
50–54	45–49		2
...			
75–80	73–78		1

the associated sunspot has died out. It is thought, therefore, that the material included in Table III is fairly representative of flocculi associated with sunspots during the period covered.

The data for eruptive flocculi are meagre at present but may be taken to indicate the order of magnitude of the enhanced emission shown by this class.

It is of interest to compare Table III with other data given in Table IV derived by the simple method * referred to previously, of a neutral wedge inserted in the field of view of the spectrohelioscope ; a wedge measure of equality affords a direct comparison between the brightness of a flocculus and that of the adjacent background at the *same* wave-length—normally that of the centre of *Hα*. This table presents the data obtained at Greenwich in this manner for normal and for eruptive flocculi for the period 1932–35 † as a whole and also for the period 1934–35, during which the new method of measuring intensities has been employed.

Table IV shows that a maximum frequency for the normal flocculi occurs at a light intensity of 1·4, *i.e.* the ratio of intensity of flocculus to that of the adjacent background at the same wave-length is 1·4. The mean of the second column (data for normal flocculi, 1934–35) is 1·46. Since the maximum intensity of the normal bright flocculi occurs within ·05 A. from the centre of *Hα*,‡ a factor of 23, taken from our contour of *Hα*, multiplied by 1·46 gives a mean value of 34 per cent. that of the continuous spectrum ; this value is in good agreement with the mean value derived quite independently from the corresponding column in Table III, viz. 36 per cent. (both figures uncorrected).

(2) *Hβ Flocculi.*—The spectrohelioscope can be fairly rapidly adjusted for observations in *Hβ*. Normally, however, *Hβ* flocculi corresponding

* *M.N.*, **93**, 172, 1933.

† It may be noted that Table VI given in the previous paper, *loc. cit.*, includes measures of light intensity of *Hα* flocculi both for areas in which sunspots were present and those in which the associated spot had died out.

‡ *M.N.*, **93**, 170, 1933.

TABLE IV

Intensities of Bright Hα Flocculi associated with Sunspots (Simple Wedge Method)

Intensity	Number of Flocculi			
	Normal		Eruptive	
	1932–35	1934–35	1932–35	1934–35
1·1	7	2		
1·2	23	17		
1·3	42	24		
1·4	59	29		
1·5	41	20		
1·6	20	9		
1·7	16	8	1	1
1·8	8	4	1	0
1·9	7	3	1	0
2·0	5	3	5	0
2·1	2	1	2	0
2·2	1	1	6	3
2·3	0	0	1	1
2·4	0	0	1	0
2·5	2	1	2	2
2·6–2·9			2	2
3·0–3·4			2	1
3·5–3·9			0	0
4·0–4·4			1	1
4·5–5·0			1	1

in place to the bright *Hα* areas are dark * and not bright, but when there are bright eruptive flocculi in *Hα* or in H or K, bright *Hβ* flocculi also appear. Future observations should decide whether this is an invariable rule of eruptive flocculi. The short life of these eruptive phenomena makes it difficult to obtain accurate comparative measures in *Hα* and *Hβ*, but it is hoped to accumulate data during the coming sunspot maximum. A measure on a small bright flocculus in *Hβ* on 1935 June 30 gave 49 per cent. that of the continuous spectrum, the corresponding measure in *Hα* being 44 per cent.

5. *Intensities of Prominences*

(1) *Prominences in Hα.*—During the period of observation, intensity measures were made of 101 prominences visible in *Hα*. The comparison of intensity was made between each prominence and the continuous spectrum 10 A. away from the centre of *Hα* at the centre of the Sun's disc. The prominences were mostly between 1′ and 2′ in height and, with the exception of five, were relatively stable in nature. The frequency of the measured intensities is indicated in Table V, the five eruptive prominences being tabulated in a separate column.

* Illustrations of this statement are given in *Publications of the Yerkes Observatory*, **3** (I), Plates 8 and 9 (1903). Characteristic phenomena visible respectively in *Hα* and *Hβ* are summarized in *Annales de l'Observatoire de Paris* (Meudon Section), **8** (II), 42, 1930.

TABLE V

Intensities of Hα Prominences Compared with the Centre of the Sun's Disc, 1934–35

Intensity		Number of Prominences	
Uncorrected	Corrected	Quiescent	Eruptive
1– 4	1– 4	10	
5– 9	5–10	51	
10–14	11–15	25	
15–19	16–21	10	1
20–24	22–26		1
25–29	27–32		1
30–34	33–37		0
35–39	38–43		1
50–60	55–66		1

The mean observed intensity of quiescent prominences summarized in Table V is 9 per cent. uncorrected (or 10 per cent. corrected) that of the continuous spectrum. The few observations that have been made of eruptive prominences (characterized by rapid changes and motions in the line of sight) indicate the greater brightness of this class of prominence. It is hoped that a large number of eruptive prominences will be measured as the present sunspot cycle develops.

As an example of the measures of intensity of an eruptive prominence, we may refer to one observed at Greenwich on 1935 May 11 at the west limb. This prominence appeared to originate from a bright eruptive flocculus which contained a sunspot passing around the limb. As the line-shifter was turned from positions indicating radial velocities of +50 to −80 km./sec. the shape and apparent brightness of the prominence changed very considerably, and the following measures of intensity were made :—

Time 1935 May 11 U.T.		Radial Velocity km./sec.	Intensity (Uncorrected)
h	m		
10	44	−45	44
10	50	0	24
10	55	−80	39
11	00	−25	18
11	15	+50	58
11	20	+50	54

The association of high emission with large radial velocity at the limb is interesting and worth future investigation.

(2) *Prominences in Hβ.*—In general, the observation of intensity of prominences in *Hβ* is difficult with the present equipment on account of their relative faintness as compared with their *Hα* images. Observations were made, however, when suitable prominences presented themselves, in order to obtain a measure of their relative intensities, *Hβ*/*Hα*, and the results of comparisons of 22 prominences are given in Table VI. These prominences were, with two exceptions, indicated with an asterisk, stable

and ranged in height from 0′·5 to 2′·5. In making the reductions it was assumed for a first approximation that the contour of $H\beta$ was similar to that of $H\alpha$.

TABLE VI

Comparison of Intensities of Prominences in Hα and Hβ, 1934–35

Date 1934	Ratio of Intensity $H\beta/H\alpha$	Date 1935	Ratio of Intensity $H\beta/H\alpha$
Aug. 26	·24	May 2	·40
28	·25	June 21	·33
31	·20	22	·31*
31	·33	22	·33
Sept. 28	·10	23	·11*
28	·12	30	·15
Oct. 4	·27	Aug. 5	·20
11	·20	5	·14
11	·28	Sept. 2	·17
12	·53	11	·40
		Oct. 23	·49
		23	·20
Mean	·25		·27

Table VI shows that the ratio of the brightness of a particular prominence in $H\beta$ to its brightness in $H\alpha$ varies from 0·10 to 0·53 (mean 0·26). The probable error in the measured brightness in $H\beta$ is of order 10 per cent., and the probable error of the measure in $H\alpha$ is less, so that we conclude that the variations in the ratio $H\beta/H\alpha$ from one prominence to another are probably real, *i.e.* the Balmer decrement varies substantially from one prominence to another. This variation is shown by measurements from $H\beta$ to $H\delta$ made by Schwarzschild on prominences which he observed at the eclipse of 1905 (*Ast. Mitt. Gottingen*, **13**, 53, Table XIX, 1906). It is hoped to obtain more information on this point by extending the observations if possible to include measurements in $H\gamma$, but visual work with the spectrohelioscope in this region of the spectrum is very restricted.

6. *Future Work with the Instrument*

It is not proposed to attempt to introduce any refinement into the measures of line contours, because it is hoped to obtain better contours with a new instrument embodying a monochromator. But the spectrohelioscope being peculiarly well adapted to the observation of transient phenomena on the disc, work on the flocculi and prominences will be continued.

The simple wedge method for comparing the intensity of a flocculus with that of the adjacent background at the same wave-length appears to give satisfactory results and is more rapid in operation—a definite advantage when the sky is variable.

In both methods we have hitherto measured the brightness of each prominence or flocculus at maximum contrast with the disc. An attempt

will be made next summer to obtain measures at other points, so as to obtain the contour of the emission line of the prominence, and that of the absorption line, either as a reversal or a filling in of the normal $H\alpha$, in the case of the flocculi. These observations may be expected to throw light on the question of the mechanism of the emission, or lack of absorption, in the areas concerned.

The bulk of the observations described in the present paper have been carried out by one of us (H. W. N.), but a number of observations have also been made by Mr. H. J. Barton.

Summary.—An attachment to the spectrohelioscope is described, by means of which intensities of bright flocculi and prominences may be referred to the intensity of the solar disc outside the line $H\alpha$ or $H\beta$. The contour of $H\alpha$ was determined with this apparatus, without photographic photometry. The most frequent intensity of non-eruptive flocculi in $H\alpha$ is 0·30 of the intensity of the neighbouring disc at wave-lengths outside $H\alpha$, eruptive flocculi being brighter. Prominences were measured in both $H\alpha$ and $H\beta$. Non-eruptive prominences in $H\alpha$ show a most frequent intensity of 0·10 of the intensity of the centre of the disc outside $H\alpha$. Eruptive prominences are much brighter. The ratio of intensity of $H\beta/H\alpha$ averages 0·26 when both are referred to the centre of the disc, but this decrement appears to vary from one prominence to another.

Royal Observatory, Greenwich :
1935 *October* 28.

THE STRUCTURE OF SUNSPOTS.

T. G. Cowling, M.A., D.Phil.

1. It is the object of the present paper to give prominence to certain points in the theory of the structure of sunspots which appear hitherto to have escaped general notice.

It is normally assumed, following H. N. Russell,* that the darkening of a sunspot is due to a rapid upward motion of gas, accompanied by an adiabatic cooling. On this hypothesis E. A. Milne † estimated that the layer from which the gas rises to the surface is probably not more than 100 km. below the photosphere : this estimate is, however, considerably to be increased if the atmosphere consists chiefly of hydrogen, as is now believed.

Milne assumed that gas is being sucked into the spot at low levels, rising to the surface and there flowing out of the spot, and then again sinking. The whole motion is a circulation which, he suggested, will in the absence of viscosity continue to exist, once started, just as liquid will under similar circumstances continue to circulate round a vortex ring.

The analogy is, however, imperfect, as it ignores the fact that variation of temperature in a gas is often more important than viscosity in limiting

* *Ap. J.*, **54**, 293, 1921. † *M.N.*, **90**, 487, 1930.

the motions set up in it. In an atmosphere in which the upward temperature gradient is less than the adiabatic gradient, gas which has risen from below is in general cooler and denser than the surrounding gas, and so tends to move downwards under gravity. Thus the natural tendency is for small motions set up in such a "stable" atmosphere to be oscillatory, not circulatory; large circulations may exist, but only if the velocities involved are comparable with the velocity of fall under gravity through a distance of the dimensions of the disturbance. Moreover, when the rates of transport of energy by convection and radiation are comparable, any motion, whether oscillatory or circulatory, will be rapidly damped, just as sound waves in the atmosphere are damped if the variation of temperature is not almost exactly adiabatic.

Milne considered a stable atmosphere and relatively small velocities: hence his circulation should not be capable of continued existence. In fact, assuming that the pressure and temperature at the base of the spot column are the same as in the surrounding matter at the same level, he found that at the surface the pressure in the spot is considerably less than that outside. This difference of pressures at the surface will, in the absence of other factors, convert his circulatory motion into oscillatory within at most a few minutes.

Milne's work, therefore, does not supply a reason for the continued existence of a spot. A possible reason was supplied by A. Unsöld,* who pointed out that in a layer just below the photosphere hydrogen is becoming ionised, and the corresponding specific heat is very large. Hence the actual temperature is greater than the adiabatic gradient, and the layer is unstable.

2. The degree of instability in the layer is increased by the fact that the radiative temperature gradient is very large here, because of the increase of opacity that follows from the ionisation of the hydrogen. To illustrate this effect, the writer has performed a numerical integration of the equations of radiative equilibrium

$$\frac{dp'}{dh} = \frac{\kappa\rho F}{c}, \qquad \frac{dp}{dh} = g\rho,$$

where p, p' are the total pressure and the radiation pressure, F is the outward flux of radiation per unit area, ρ the density, κ the opacity, c the velocity of light, g the acceleration of gravity, and h the depth below a suitably defined photosphere. The opacity formula used is that of Chandrasekhar.† The atmosphere is supposed to be a mixture of hydrogen and a more readily ionised gas of ionisation potential 8 volts: one atom in fifty is supposed to belong to the second gas, and the mean atomic weight of the mixture is taken to be 2. The initial conditions adopted are that the temperature T is 5750° K., and that $p = 5000$ dynes/cm.^{2}, when $h = 0$; this corresponds to an electron pressure of order 100 dynes/cm.^{2}. The contribution of radiation pressure to the support is, for simplicity, neglected.

* *Zeit. für Astrophys.*, **1**, 138, 1930; **2**, 209, 1931.

† *Proc. Roy. Soc.*, **135**, 472, 1932. It must be emphasised that the phenomenon does not depend on the particular opacity formula used.

The results of the integration are set out in the table. From this we see that in the surface layers the temperature gradient is about 15°/km. ; in very deep layers, corresponding to the reduction in mean molecular weight due to the ionisation of the hydrogen, it is reduced to about half this ; but in the intermediate layers it is much greater, attaining a maximum value of about 240°/km. between $h = 157$ km. and $h = 166$ km. Between $h = 142$ km. and $h = 221$ km. the density actually diminishes as the depth increases. Thus conditions in this layer are very favourable to the production of convection.

h in km.	p in dynes/cm.^{2}	ρ in grs./cm.^{3}	T	$\varkappa$
0	5×10^3	$2{\cdot}051 \times 10^{-8}$	5750° K.	·00001
44	$8{\cdot}174 \times 10^3$	$2{\cdot}646 \times 10^{-8}$	6470°	·0002
95	$1{\cdot}039 \times 10^4$	$4{\cdot}271 \times 10^{-8}$	7259°	·0020
125	$1{\cdot}689 \times 10^4$	$4{\cdot}989 \times 10^{-8}$	7914°	·0090
142	$1{\cdot}924 \times 10^4$	$5{\cdot}116 \times 10^{-8}$	8628°	·0298
157	$2{\cdot}135 \times 10^4$	$3{\cdot}862 \times 10^{-8}$	11180°	·3659
166	$2{\cdot}196 \times 10^4$	$2{\cdot}177 \times 10^{-8}$	13280°	·8235
221	$2{\cdot}478 \times 10^4$	$1{\cdot}792 \times 10^{-8}$	16730°	·9879
407	$3{\cdot}473 \times 10^4$	$2{\cdot}104 \times 10^{-8}$	19880°	·9984
1001	$8{\cdot}318 \times 10^4$	$4{\cdot}000 \times 10^{-8}$	25020°	·9999
1911	$2{\cdot}380 \times 10^5$	$9{\cdot}090 \times 10^{-8}$	31510°	1·0000
3760	$1{\cdot}210 \times 10^6$	$3{\cdot}271 \times 10^{-7}$	44500°	1·0000
5399	$3{\cdot}578 \times 10^6$	$7{\cdot}685 \times 10^{-7}$	56030°	1·0000
7454	$1{\cdot}053 \times 10^7$	$1{\cdot}797 \times 10^{-6}$	70530°	1·0000

3. Observation reveals that the actual temperature gradient in the surface layers of a sunspot is approximately the radiative gradient corresponding to the effective temperature of the spot.* It has been suggested that this establishes that any circulation connected with a spot must be limited to deep layers : but actually the existence of large mass motions in the surface layers does not necessarily imply a serious departure from radiative equilibrium there.

Suppose for simplicity that the velocity w of gas in a sunspot is in the upward direction, which is taken to be the direction of Oz, and that the effect of a horizontal flow of radiation into the spot can be neglected. Then, if E is the thermal energy per unit mass, the equation of thermal energy is †

$$\frac{\partial E}{\partial t} + w\frac{\partial E}{\partial z} = \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t} + w\frac{\partial \rho}{\partial z}\right) + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{c}{\kappa\rho}\frac{\partial p'}{\partial z}\right), \qquad (1)$$

or, using the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho w) = 0$$

is

$$\frac{\partial E}{\partial t} = -w\frac{\partial E}{\partial z} - \frac{p}{\rho}\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{c}{\kappa\rho}\frac{\partial p'}{\partial z}\right).$$

* Minnaert and Wanders, *Zeit. für Astrophys.*, **5**, 297, 1932.

† This energy equation can readily be derived from the general one given by L. H. Thomas (*Quart. Journ. Math.* (Oxford), **1**, 239, 1930) on replacing his D/Dt by $\partial/\partial t + w\partial/\partial z$, and simplifying.

The first two terms on the right of this represent the effect of convection, the third that of radiation.

Let c_v be defined by the equation

$$\frac{\partial E}{\partial z} = c_v \frac{\partial T}{\partial z}$$

so that c_v becomes equal to the specific heat per unit mass at constant volume when the effect of ionisation and radiation can be ignored. Then c_v is not less than $3p/2\rho T$, its value for a monatomic gas ; hence the terms in the equation of energy giving the effect of convection are of the same order as

$$\frac{1}{\rho} \frac{\partial}{\partial z}(\rho c_v T w).$$

Thus, roughly speaking, convection is only more important than radiation if

$$\rho c_v T |w| > \frac{c}{\kappa\rho} \left| \frac{\partial p'}{\partial z} \right| .$$

The expression on the right of this inequality is the upward flux of radiation which we take as a quarter of the mean flux over the whole disc. This gives

$$\rho c_v T |w| > 1{\cdot}5 \times 10^{10} \text{ grs./sec.}^3.$$

Above or below the region in which hydrogen becomes ionised, $\rho c_v T = \frac{3}{2} p$ approximately. Hence, setting $|w| = 10^5$ cm./sec., of the order of the observed surface velocities, we find that convection only dominates radiation if $p > 10^5$ dynes/cm.^{2}. This corresponds to a layer considerably below the reversing layer, unless the proportion of hydrogen in the atmosphere is inordinately large.* It follows that the circulation associated with a spot can extend to visible layers and yet the temperature gradient be approximately radiative there.

In the region in which hydrogen becomes ionised, the energy per unit volume (neglecting the small effect of radiation and of atoms other than hydrogen) is $\frac{3}{2} p + nx\chi$, where n is the number of hydrogen atoms per unit volume, x is the proportion ionised, and χ is the energy of ionisation. Hence, remembering that $p \propto n(1+x)T$, we find that

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* A result similar to this was obtained for small-scale convection by Siedentopf, *A.N.*, **247**, 297, 1932. Siedentopf's estimate of turbulent velocities in the atmosphere requires correction for two reasons : first, he assumes that in a "mischungs-weg" the temperature of a moving mass of gas varies adiabatically, ignoring the effect of radiation : secondly, he does not observe the possibility of the large increase in temperature gradient which we have shown to occur in the region in which hydrogen becomes ionised. As these two factors act in opposite directions, his estimate that the turbulent velocities are of order 1 km./sec. is probably not wide of the truth. By similarly equating the work done on gas in a sunspot with the energy dissipated by eddies it should be possible, when the theory of the atmosphere is more complete, to determine theoretically the velocity of gas in a sunspot.

$$c_v\frac{\partial T}{\partial z}=\frac{\partial}{\partial z}[(\tfrac{3}{2}p+nx\chi)/\rho]$$

$$=\tfrac{3}{2}\frac{p}{\rho T}\frac{\partial T}{\partial z}+\left(\frac{n\chi}{\rho}+\frac{3p}{2\rho(1+x)}\right)\frac{\partial x}{\partial z}.$$

For example, in the table given above x increases from 0·37 to 0·82 as h increases from 157 km. to 166 km., and T increases from 11180° to 13280°; this corresponds to an average value of about $25p/\rho T$ for c_v. If we suppose that a similar value also obtains in the spot column—the actual value is somewhat smaller—we find that with $w=10^5$ cm./sec. convection predominates if $p>7\times10^3$ dynes/cm.^{2}, which is true at this depth. Hence it would appear possible for motion at lower levels to result in the observed spot darkening, while the temperature gradient near the surface remains approximately radiative.

4. Now, however, another difficulty appears. Equation (1) can be written in the form

$$\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{c}{\kappa\rho}\frac{\partial p'}{\partial z}\right)=\frac{\partial E}{\partial t}+p\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)+w\left(\frac{\partial E}{\partial z}+p\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\right).$$

Since conditions in a spot vary slowly, the first two terms on the right-hand side of this equation can be neglected. The equation can then be interpreted as showing that the outward flux of radiation only diminishes with increasing height (as must happen in the region below a sunspot) if

$$w\left(\frac{\partial E}{\partial z}+p\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\right)$$

is positive. Now the expression in the bracket in this represents the result of subtracting the actual gradient of the energy per unit mass from the corresponding adiabatic gradient, and is positive or negative according as the atmosphere is stable or unstable. Hence, if the observed darkening of the spot is due to a departure from radiative equilibrium in the unstable region of § 2, w must be negative there; that is, the motion of the gas must be downward.

Observation, on the other hand, indicates that the motion of the surface layers in a spot is upward.* If the motion in lower layers is also upward, it can only produce the observed darkening of the spot if it extends to very deep layers, where the atmosphere is stable, and if the effect of the unstable region is relatively unimportant: in this case, by the argument of § 1, some force other than the ordinary mechanical forces is required to maintain the motion. If no such force exists—and there seems no reason why it should—motion in the lower layers must be downward, and be confined to the unstable region and its neighbourhood. Hence it seems necessary to suppose that there is a difference between the motion of the surface layers and that of those immediately below them. It is hard to believe that this

* St. John, *Ap. J.*, **37**, 322, 1913.

difference could exist without showing itself by observable signs. Further observations of the flow of gas in sunspots would appear desirable.

It may be pointed out that if the motion of gas in a spot is downward it is not possible to explain the magnetic field of sunspots along the lines suggested in an earlier paper.* Magnetic fields would, instead, be associated with the bright spots that result from an upward motion.

Summary.—There is a region below the photosphere of the sun in which the temperature gradient is much greater than in the surface layers. The temperature gradient in the matter immediately below a sunspot will only diverge seriously from the radiative value in this region. To produce a dark spot, matter in this region must be moving downward.

Swansea :
1935 *October* 23.

NOTE ON "RELATIVISTIC DEGENERACY."

Sir A. S. Eddington, F.R.S.

In recent papers † I have contended that the "relativistic" degeneracy formula is erroneous. This has led Møller and Chandrasekhar ‡ to publish a note defending it. They give a derivation of the formula which is doubtless more up to date than those which I criticised. It therefore seems desirable that I should amplify my attack on the formula by showing why I am unable to accept Møller and Chandrasekhar's proof.

The objection arises almost at the beginning. Near the middle of p. 674 we read :

"To go over to the case of N electrons in a finite volume V which satisfy the exclusion principle. . . . Let $\psi_{\mathbf{p},s}$ be a suitably normalised eigen solution . . . with a definite value for the momentum $\mathbf{p}$ and spin s. Since the electrons are confined in a finite volume, $\mathbf{p}$ takes on only discrete values."

Thus at the outset they contradict the Uncertainty Principle, which asserts that when the position of an electron is known within finite limits the momentum $\mathbf{p}$ has *not* a definite value.

This is not a trivial contradiction. The consecutive discrete values which they assign to p_x differ by h/l_x ; this is precisely the order of magnitude of the uncertainty of p_x due to the x-co-ordinate of the electron being known to within a range l_x. Their first reference to *definite* values might have been left uncriticised, on the supposition that "definite" was not meant quite literally but allowed the small haziness h/l_x which wave mechanics demands. Their insistence on *discrete* values with gaps of h/l_x excludes such an interpretation.

The authors started by considering N plane Dirac waves, each representing only one electron distributed with even probability through infinite

* *M.N.*, **94**, 39, 1933.
† *M.N.*, **95**, 194, 1935 ; *Proc. Roy. Soc.*, A, **152**, 253, 1935.
‡ *M.N.*, **95**, 673, 1935.

space. One naturally wonders how they will manage to pass from this formulation to the actual problem of N electrons concentrated in a volume V. Clearly extensive modifications of the wave function are required to collect the probability into the assigned volume, and to convert the dispersing system into a steady distribution of density. The authors are silent about this. Instead, they impose (or emphasise) the very condition which, according to wave mechanics, will prevent the electrons from being within the assigned volume.

My case against former proofs was that they conflict with relativity theory ; my case against this proof is that it conflicts with wave mechanics.

The Exclusion Principle was first enunciated for electron orbits or states in an atom ; for applications of this type it has been abundantly verified. Undoubtedly there exists a generalisation of it applicable to the statistics of large assemblies of particles. But the generalisation cannot be of the form assumed by Møller and Chandrasekhar, which conflicts with the Uncertainty Principle. In my treatment (and, I think, in most earlier treatments) the generalisation is supposed to apply to standing waves. For standing waves the unit solutions correspond to discrete values of $\mathbf{p}^2$; and $\mathbf{p}$ has no definite value—as the Uncertainty Principle insists.

Møller and Chandrasekhar say that they could just as well have used standing waves in their investigation. This seems to be an error, for they assume in their analysis that the operator $\mathbf{p}$ reduces to an eigen value.

THE RADIATIVE EQUILIBRIUM OF THE OUTER LAYERS OF A STAR WITH SPECIAL REFERENCE TO THE BLANKETING EFFECT OF THE REVERSING LAYER.

S. Chandrasekhar, Ph.D.

1. *Introduction.*—For grey material in radiative equilibrium we have Milne's well-known result that

$$B(\tau) = \tfrac{1}{2}F(1 + \tfrac{3}{2}\tau), \tag{1}$$

where $B(\tau)$ is the intensity of the integrated black-body radiation for the temperature at the optical depth τ and πF is the constant net flux. From (1) we have for the specific intensity $I(\theta)$ of the emergent radiation in a direction making an angle θ with the normal, the expression

$$I(\theta) = \int_0^\infty B(\tau \cos\theta) e^{-\tau} d\tau = \tfrac{1}{2}F(1 + \tfrac{3}{2}\cos\theta). \tag{2}$$

(2) corresponds to a coefficient of darkening $u = 3/5$. We shall refer to (2) as the *standard darkening*.

Again (1) leads to a definite prediction concerning the intensity distribution in the continuous spectrum. Thus the intensity $I(\lambda, \theta)$ of the emergent

radiation at wave-length λ and in a direction making an angle θ with the normal is given by

$$I(\lambda) = \int_0^\infty B(\lambda, T_{\tau \cos\theta}) e^{-\tau} d\tau, \tag{3}$$

where $T_{\tau \cos\theta}$ is the temperature at optical depth $\tau \cos\theta$ and $B(\lambda, T)$ the Planck function corresponding to the temperature T. The comparison between the predicted intensity distribution in the continuous spectrum with the observed distribution in the case of the Sun has been examined by Milne * and Lindblad.† The surprising result emerged that the predictions of the theory were in much better accord with the results of observation than one had reason to expect. This fair agreement between the theory and the observations has generally been taken to mean that the overlapping of a vast number of photo-electric absorption bands from a large number of different elements in their various excited states does in fact produce an approximate " greyness " of the Sun's material.

2. However, the assumption $\kappa = \text{constant}$ still leaves certain significant discordances between theory and observation. Thus a coefficient of darkening $u = 0{\cdot}56$ agrees better with the results of observation. Further, as H. H. Plaskett's observations showed, the true colour temperature of the Sun at the centre of the disc is much higher than could be expected on the basis of (3). It is therefore necessary to examine whether the taking into account of the formation of the absorption lines in the higher levels of the photosphere could bring about a better agreement between the theory and observations. An obvious effect of the formation of the Fraunhofer lines in the " reversing layer " is to keep the photosphere warmer, since the energy dammed back in the Fraunhofer lines is necessarily returned to the Sun's photosphere. The reversing layer therefore effectively acts as a blanket. In his classical memoir on the subject Milne examines this blanketing effect by considering a purely scattering atmosphere placed above grey material in radiative equilibrium. Subsequently Lindblad gave a formally equivalent treatment of this problem in which he assumed that the distribution of temperature in the photosphere is unaltered, but supposed it to terminate at some fictitious optical depth $\tau = \tau_0$. At $\tau = \tau_0$ the outward flux πF_+ is found to be

$$\pi F_+ = \pi F(1 + \tfrac{3}{4}\tau_0). \tag{4}$$

The inward flux then is

$$\pi F_- = \pi F(1 - \tfrac{3}{4}\tau_0). \tag{4'}$$

The assumption is now made that

$$F_- = \eta F_+, \tag{5}$$

where η is non-zero. (5) is usually stated in the following terms: "The reversing layer returns to the photosphere a fraction η of the radiation incident on it from the photosphere." However, the physical assumptions

* *Phil. Trans. Roy. Soc.*, A, **223**, 209, 1922.

† *Nova Acta Reg. Soc. Sc.*, Upsala, **6**, No. 1, p. 17, 1923.

involved are not quite clear, and as Professor H. H. Plaskett has kindly pointed out to the writer, Milne's earlier treatment is preferable in this respect. The following is Plaskett's version of Milne's treatment:—

For the temperature distribution assume that

$$B(\tau) = a + \tfrac{3}{4}F\tau, \tag{6}$$

where a is left arbitrary for the present. (That the coefficient of τ should always be $\frac{3}{4}$F is known from the theory of radiative equilibrium.*) From (6) we now find that at $\tau = 0$

$$F_+ = a + \tfrac{1}{2}F\,; \qquad F_- = a - \tfrac{1}{2}F. \tag{7}$$

If there had been no scattering atmosphere then $F_+ = F$ and $F_- = 0$. But the effect of the purely scattering atmosphere is to return to the photosphere a finite inward flux. This inward flux is mainly in the absorption lines, but it would be absorbed in the deeper photospheric layers and then converted into general radiation. We now assume that the inward flux F_- is a fraction η of the outward flux F_+. This condition determines a. We have finally

$$B(\tau) = \tfrac{1}{2}F\left[\frac{1+\eta}{1-\eta} + \tfrac{3}{2}\tau\right]. \tag{8}$$

From (8) we have the relation

$$T_0^4 = \tfrac{1}{2} \cdot \frac{1+\eta}{1-\eta} T_e^4. \tag{9}$$

Again from (8) and (2) we have

$$I(\theta) = \tfrac{1}{2}F\left[\frac{1+\eta}{1-\eta} + \tfrac{3}{2}\cos\theta\right]. \tag{10}$$

(10) corresponds to a coefficient of darkening

$$u = \frac{3(1-\eta)}{5-\eta}. \tag{11}$$

As the observed coefficient of darkening is nearer 0·56, it would, according to (11), correspond to a value of $\eta = 0{\cdot}083$.†

The modified intensity distribution in the continuous spectrum can be found by inserting (8) in (3). Such calculations have been made by Milne and Lindblad. Thus Lindblad finds that the detailed predictions of the theory for the darkening at each separate wave-length are fairly confirmed by observations.

In spite, however, of the formal success of the Milne-Lindblad treatment of the blanketing effect of the reversing layer, it is clear that a satisfactory theory ought really take into account the formation of the absorption lines in a much more direct way than has so far been done. In this paper an attempt in this direction is made. We shall assume in particular that *the absorption lines are uniformly distributed in the whole spectral range*, and then

* Milne, *M.N.*, **81**, 367, 1921.

† See G. F. W. Mulders, "Aequivalente Breedten van Fraunhoferlijnen in het Zonnenspectrum" (Doctor Thése, Utrecht).

calculate the altered temperature distribution and also the complete march of the integrated fluxes in the absorption lines and in the continuous spectrum. We shall also calculate the modified law of darkening and consider the circumstances under which the theory can predict a reduced centre-limb contrast in agreement with observation. We shall also estimate the true colour temperatures on the basis of our present more exact analysis of the radiative equilibrium of the outer layers. It is found that Plaskett's high value for the true colour temperature of the Sun can be accounted for theoretically as due to the blanketing effect of the reversing layer.

4. Before concluding this introductory section reference should be made to a recent paper by Woolley,* where the blanketing effect of the reversing layer is taken into account on essentially the same lines as we do. However, our analysis is much more general and Woolley's calculations appear as a very special case.

Section I

5. As stated in § 3 we shall assume that the absorption lines occur uniformly in the whole spectral range. This statement should be made more precise. Let $a_2(\nu)d\nu$ be the probability of our finding in the frequency interval $(\nu, \nu+d\nu)$ an absorption line. The probability then that the interval is free from an absorption line is

$$a_1(\nu)d\nu = \{1 - a_2(\nu)\}d\nu. \tag{12}$$

Our assumption now is that

$$a_1(\nu) = a_1 = \text{constant}, \tag{13}$$

$$a_2(\nu) = a_2 = \text{constant}. \tag{13'}$$

It is of course clear that $a_1 + a_2 = 1$.

6. *The Equations of Transfer.*—Let $I_\nu d\nu$ be the specific intensity of the radiation in the frequency interval $(\nu, \nu+d\nu)$. Then if we are outside an absorption line the equation of transfer (on the hypothesis of local thermodynamic equilibrium) is as usual

$$\cos\theta\frac{dI_\nu}{\rho dx} = -\kappa_\nu I_\nu + \kappa_\nu B_\nu, \tag{14}$$

where κ_ν the coefficient of absorption and $B_\nu d\nu$ the intensity of the black-body radiation at the temperature at x and in the frequency interval $(\nu, \nu+d\nu)$.

On the other hand, if we are inside an absorption line, then the appropriate equation of transfer is

$$\cos\theta\frac{dI_\nu}{\rho dx} = -(\kappa_\nu + a_\nu)I_\nu + (1-\epsilon_\nu)a_\nu \mathfrak{J}_\nu + (\kappa_\nu + \epsilon a_\nu)B_\nu, \tag{15}$$

where

$$\mathfrak{J}_\nu = \tfrac{1}{2}\int_0^\pi I_\nu \sin\theta d\theta, \tag{16}$$

* R. v. d. R. Woolley, *M.N.*, **94**, 713, 1934. An error in his formula rather spoils the numerical work in this paper (see § 13 of the present paper, especially the footnote on p. 29).

and κ_ν (as in (14)) is the coefficient of continuous absorption and α_ν the *mean* coefficient of scattering in the line. Further, ϵ_ν is the probability that the radiation at $(\nu, \nu+d\nu)$ is lost by photoelectric ionizations and also by collisions of the second kind.*

We shall now assume that outside the absorption lines $\kappa_\nu = \kappa = \text{constant}$. We shall also assume that the mean scattering coefficient α_ν in the different absorption lines and the probability coefficients ϵ_ν are all the same and are respectively α and ϵ (say).

Now integrate the equation of transfer (14) over the continuous spectrum only. By (13) then

$$\cos\theta\frac{dI_1}{\rho dx} = -\kappa(I_1 - a_1 B), \tag{17}$$

where I_1 is the intensity of the integrated radiation in the continuous spectrum and B now is the intensity of the integrated black-body radiation at x. If I_2 is the intensity of the integrated radiation in the different absorption lines, then we have

$$\cos\theta\frac{dI_2}{\rho dx} = -(\kappa+\alpha)I_2 - (1-\epsilon)\alpha\mathcal{J}_2 + a_2(\kappa+\epsilon\alpha)B. \tag{18}$$

Notice the appearance of the probability factors a_1 and a_2 in front of B in the equations (17) and (18). ($\mathcal{J}_2$ is defined below.)

It may be noticed here that *if the absorption lines are formed in "pure" local thermodynamic equilibrium, then* $\epsilon = 1$. On the other hand, *if the absorption lines are formed as a pure scattering phenomenon, then* $\epsilon = 0$.

7. We shall introduce the following quantities :—

$$\mathcal{J}_1 = \tfrac{1}{2}\int_0^\pi I_1 \sin\theta d\theta\,; \qquad \mathcal{J}_2 = \tfrac{1}{2}\int_0^\pi I_2 \sin\theta d\theta,$$

$$F_1 = 2\int_0^\pi I_1 \sin\theta\cos\theta d\theta\,; \qquad F_2 = 2\int_0^\pi I_2 \sin\theta\cos\theta d\theta,$$

$$K_1 = \tfrac{1}{2}\int_0^\pi I_1 \sin\theta\cos^2\theta d\theta\,; \qquad K_2 = 2\int_0^\pi I_2 \sin\theta\cos^2\theta d\theta.$$

On multiplying the equations of transfer successively by $\frac{1}{2}\sin\theta d\theta$ and $\frac{1}{2}\sin\theta\cos\theta d\theta$ and integrating from 0 to π we find that

$$\tfrac{1}{4}\frac{dF_1}{\rho dx} = -\kappa(\mathcal{J}_1 - a_1 B), \tag{19}$$

$$\tfrac{1}{4}\frac{dF_2}{\rho dx} = -(\kappa+\epsilon\alpha)(\mathcal{J}_2 - a_2 B), \tag{19'}$$

and

$$\frac{dK_1}{\rho dx} = -\tfrac{1}{4}\kappa F_1\,; \qquad \frac{dK_2}{\rho dx} = -\tfrac{1}{4}(\kappa+\alpha)F_2. \tag{20}$$

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* For a precise derivation of (15) see an important paper by B. Strömgren, *Zeits. für Astrophysik*, **10**, 237, 1935.

8. *The Equation of Radiative Equilibrium and the Constancy of Net Flux.* —The equation of radiative equilibrium is obtained by equating the total absorption and the total emission in an element of volume. By an application of Kirchoff's law this is easily to be

$$\kappa \mathcal{J}_1 + (\kappa + \epsilon\alpha)\mathcal{J}_2 = (a_1\kappa + a_2(\kappa + \epsilon\alpha))B. \tag{21}$$

Adding equations (19) and (19′) and using (21) we have

$$\frac{dF_1}{dx} + \frac{dF_2}{dx} = 0,$$

or

$$F_1 + F_2 = \text{constant} = \mathfrak{F} \text{ (say)}, \tag{22}$$

—*i.e.* the integrated total net flux is constant.

9. We shall now assume that

$$\frac{\alpha}{\kappa} = \text{constant} = \xi \text{ (say)}. \tag{23}$$

Introducing now the optical depth τ defined by

$$d\tau = -\kappa\rho dx, \tag{24}$$

the equations (19) and (20) now reduce to

$$\tfrac{1}{4}\frac{dF_1}{d\tau} = \mathcal{J}_1 - a_1 B\,; \qquad \tfrac{1}{4}\frac{dF_2}{d\tau} = \lambda(\mathcal{J}_2 - a_2 B) \tag{25}$$

and

$$\frac{dK_1}{d\tau} = \tfrac{1}{4}F_1\,; \qquad \frac{dK_2}{d\tau} = \tfrac{1}{4}\mu F_2, \tag{26}$$

where λ and μ are defined as

$$\lambda = 1 + \epsilon\xi\,; \qquad \mu = 1 + \xi. \tag{27}*$$

From the equations (26) we now have

$$\frac{dK_1}{d\tau} + \frac{1}{\mu}\frac{dK_2}{d\tau} = \tfrac{1}{4}(F_1 + F_2) = \tfrac{1}{4}\mathfrak{F} \tag{28}$$

or

$$K_1 + \frac{1}{\mu}K_2 = \tfrac{1}{4}\mathfrak{F}\tau + c, \tag{29}$$

where c is an integration constant. We shall refer to (29) as the *K-integral.*

10. We introduce now the usual approximation

$$\mathcal{J} = 3K. \tag{30}$$

Then from the equation of radiative equilibrium

$$B = \frac{3}{a_1 + a_2\lambda}(K_1 + \lambda K_2), \tag{31}$$

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* In "pure" local thermodynamical equilibrium $\epsilon = 1$ and $\lambda = \mu$. For the case where absorption lines arise as a pure scattering phenomenon $\epsilon = 0$ and $\lambda = 1$.

or using the K-integral (we have on eliminating K_1)

$$B=\frac{3}{a_1+a_2\lambda}\left[(\tfrac{1}{4}\mathfrak{H}\tau+c)+\left(\lambda-\frac{1}{\mu}\right)K_2\right], \tag{32}$$

or alternatively eliminating K_2,

$$B=\frac{3}{a_1+a_2\lambda}[\lambda\mu(\tfrac{1}{4}\mathfrak{H}\tau+c)-(\lambda\mu-1)K_1]. \tag{33}$$

From (25) we now have

$$\frac{d^2K_1}{d\tau^2}=3K_1-a_1B$$

or using (33)

$$\frac{d^2K_1}{d\tau^2}=\frac{3\lambda(a_1\mu+a_2)}{a_1+a_2\lambda}K_1-\frac{3a_1\lambda\mu}{a_1+a_2\lambda}(\tfrac{1}{4}\mathfrak{H}\tau+c). \tag{34}$$

Similarly

$$\frac{d^2K_2}{d\tau^2}=\frac{3\lambda(a_1\mu+a_2)}{a_1+a_2\lambda}K_2-\frac{3a_2\lambda\mu}{a_1+a_2\lambda}(\tfrac{1}{4}\mathfrak{H}\tau+c). \tag{35}$$

The solutions of (34) and (35) are seen to be

$$K_1=ae^{-q\tau}+\frac{a_1\mu}{a_1\mu+a_2}(\tfrac{1}{4}\mathfrak{H}\tau+c), \tag{36}$$

$$K_2=be^{-q\tau}+\frac{a_2\mu}{a_1\mu+a_2}(\tfrac{1}{4}\mathfrak{H}\tau+c), \tag{37}$$

where a and b are two integration constants and

$$q^2=\frac{3\lambda(a_1\mu+a_2)}{a_1+a_2\lambda}. \tag{38}$$

In writing (36) and (37) we have neglected a part of the solution which increases exponentially with τ. To satisfy the K-integral we must have

$$b=-a\mu. \tag{39}$$

Hence finally

$$K_1=\frac{a_1\mu}{a_1\mu+a_2}(\tfrac{1}{4}\mathfrak{H}\tau+c)+ae^{-q\tau}, \tag{40}$$

$$K_2=\frac{a_2\mu}{a_1\mu+a_2}(\tfrac{1}{4}\mathfrak{H}\tau+c)-a\mu e^{-q\tau}, \tag{41}$$

and

$$B=\frac{3\mu}{a_1\mu+a_2}(\tfrac{1}{4}\mathfrak{H}\tau+c)-\frac{3a(\lambda\mu-1)}{a_1+a_2\lambda}e^{-q\tau}. \tag{42}$$

11. From the equations (26) and (40) and (41) we obtain

$$F_1=\frac{a_1\mu}{a_1\mu+a_2}\mathfrak{H}-4aqe^{-q\tau}, \tag{43}$$

and

$$F_2 = \frac{a_2}{a_1\mu + a_2}\mathfrak{F} + 4aqe^{-q\tau}. \tag{44}$$

From (43) and (44) we see that F_1 and F_2 tend to constant limiting values as $\tau \to \infty$. Thus

$$\frac{F_1}{F_2} \to \frac{a_1}{a_2}\mu, \qquad (\tau \to \infty) \tag{45}$$

which is what we should have expected.

12. It now remains to determine the two constants of integrations a and c. The boundary conditions one adopts is to set

$$F(0) = 6K(0). \tag{46}$$

From (40), (41), (43) and (44) we have then

$$(6+4q)a + \frac{6a_1\mu}{a_1\mu + a_2}c - \frac{a_1\mu}{a_1\mu + a_2}\mathfrak{F} = 0, \tag{47}$$

$$(6\mu + 4q)a - \frac{6a_2\mu}{a_1\mu + a_2}c + \frac{a_2}{a_1\mu + a_2}\mathfrak{F} = 0. \tag{48}$$

Subtracting (48) from (47) we have the useful relation

$$\frac{\mu}{a_1\mu + a_2}c - (\mu - 1)a = \tfrac{1}{6}\mathfrak{F}. \tag{49}$$

Solving the equations (47) and (48) for a and c we find that

$$a = \frac{a_1a_2(\mu - 1)}{(a_1\mu + a_2)[6(a_1\mu + a_2) + 4q]}\mathfrak{F}, \tag{50}$$

$$c = \frac{6(a_1\mu^2 + a_2) + 4q(a_1\mu + a_2)}{6\mu[6(a_1\mu + a_2) + 4q]}\mathfrak{F}. \tag{51}$$

Inserting the above values of a and c in (42), (43) and (44) we have finally

$$B = \frac{3\mu}{a_1\mu + a_2}\left[\tfrac{1}{4}\tau + \frac{6(a_1\mu^2 + a_2) + 4q(a_1\mu + a_2)}{6\mu(6(a_1\mu + a_2) + 4q)}\right]\mathfrak{F} - \frac{3a_1a_2(\lambda\mu - 1)(\mu - 1)}{(a_1 + a_2\lambda)(a_1\mu + a_2)(6(a_1\mu + a_2) + 4q)}\mathfrak{F}e^{-q\tau} \tag{52}$$

$$F_1 = \frac{a_1\mu}{a_1\mu + a_2}\left[1 - \frac{4a_2(1 - \mu^{-1})qe^{-q\tau}}{6(a_1\mu + a_2) + 4q}\right]\mathfrak{F}, \tag{53}$$

$$F_2 = \frac{a_2}{a_1\mu + a_2}\left[1 + \frac{4a_1(\mu - 1)qe^{-q\tau}}{6(a_1\mu + a_2) + 4q}\right]\mathfrak{F}. \tag{54}$$

13. *Discussion of the Solution.*—From (52) it now follows that as $\tau \to \infty$ we have

$$B \sim \tfrac{3}{4}\frac{\mu}{a_1\mu + a_2}\mathfrak{F}\tau. \tag{55}$$

Now the Rosseland mean absorption coefficient $\bar{\kappa}$ on our assumptions is clearly given by

$$\frac{1}{\bar{\kappa}} = \frac{a_1}{\kappa} + \frac{a_2}{\kappa + \alpha}, \tag{56}$$

or

$$\kappa = \left(a_1 + \frac{a_2}{\mu}\right)\bar{\kappa}. \tag{56'}$$

Hence (55) can be rewritten as

$$B \sim \tfrac{3}{4}\mathfrak{F}\bar{\tau}, \qquad (\tau \to \infty) \tag{57}$$

where $\bar{\tau}$ is the optical depth measured in terms of the Rosseland mean absorption coefficient. (57) is an agreement with a theorem due to Milne that for material in radiative equilibrium and stratified in parallel planes, *in the far interior B increases linearly with $\bar{\tau}$ with the gradient $\frac{3}{4}\mathfrak{F}$.*

Again we notice that when either $a_2 = 0$ or $\mu = 1$, (52) reduces to

$$B = \tfrac{1}{2}\mathfrak{F}(1 + \tfrac{3}{2}\tau). \tag{58}$$

On the other hand, if $a_1 = 0$,

$$B = \tfrac{1}{2}\mathfrak{F}(1 + \tfrac{3}{2}\mu\tau). \tag{59}$$

(58) and (59) are necessary for consistency. Now, on the other hand, if $\mu \to \infty$ we have

$$B = \frac{1}{2a_1}F_1(1 + \tfrac{3}{2}\tau), \tag{60}$$

as in this case $F_2 = 0$ and $F_1 = \mathfrak{F}$.*

14. *The Boundary Temperature.*—From (52) we now find that at $\tau = 0$

$$B(0) = \tfrac{1}{2}\mathfrak{F} \cdot \frac{3(a_1\mu + a_2\lambda) + 2q(a_1 + a_2\lambda)}{(a_1 + a_2\lambda)[3(a_1\mu + a_2) + 2q]}. \tag{61}$$

In "pure" thermodynamic equilibrium $\epsilon = 1$ and $\lambda = \mu$, in which case (61) can be expressed alternatively as

$$B(0) = \tfrac{1}{2}\mathfrak{F}\frac{2q(a_1 + a_2\lambda) + 3\lambda}{2q(a_1 + a_2\lambda) + 3\lambda + 3a_1a_2(\lambda - 1)^2}. \tag{62}$$

(62) shows that the boundary temperature is *less* than what it would be if there were no absorption lines. This result apparently contradicts the expectation that the effect of the formation of the absorption lines is to keep the photosphere warmer. There is, however, no real contradiction here. Our earlier result (57) shows that in the interior the temperature

* Woolley, in his paper already referred to, considers precisely the limiting case $\mu \to \infty$. But the solution he uses is seen to be

$$B = \tfrac{1}{2}F_1\left(1 + \frac{3}{2a_1}\tau\right), \tag{60'}$$

which is in contradiction with our (60). One easily convinces oneself that "a_1" should occur as it does in (60) and not as in the above equation. This error in his formula, which appears to arise from the method of approximation adopted by Woolley, rather detracts from the value of his numerical work.

does increase with a larger gradient. But near the boundary it becomes relatively cooler. We shall come back to this point later.

If the absorption lines are formed as a pure scattering phenomenon, then $\epsilon = 0$ and $\lambda = 1$, and (61) now shows that

$$B(0) \equiv \tfrac{1}{2}\mathfrak{F} \tag{63}$$

—*i.e.* the formation of the absorption lines in this case has no effect on the boundary temperature.

15. *The Law of Darkening. Case* (*a*) $\lambda = \mu$.—We shall first consider the case of pure local thermodynamical equilibrium, *i.e.* $\lambda = \mu$.

Let $I_1(\theta)$ and $I_2(\theta)$ be the intensities of the integrated emergent radiations in a direction making an angle θ with the normal in the continuous spectrum and in the absorption lines respectively. Then clearly

$$I_1(\theta) = a_1 \int_0^\infty B(\tau \cos\theta) e^{-\tau} d\tau, \tag{64}$$

$$I_2(\theta) = a_2 \int_0^\infty B\left(\frac{\tau}{\mu} \cos\theta\right) e^{-\tau} d\tau. \tag{65}$$

Using the expression for B given in equation (42) we have

$$I_1(\theta) = \frac{3a_1\mu}{a_1\mu + a_2}\left(\tfrac{1}{4}\mathfrak{F} \cos\theta + c\right) - \frac{3a_1(\mu^2 - 1)}{(a_1 + a_2\mu)(1 + q\cos\theta)} a, \tag{66}$$

$$I_2(\theta) = \frac{3a_2\mu}{a_1\mu + a_2}\left(\frac{1}{4\mu}\mathfrak{F} \cos\theta + c\right) - \frac{3a_2(\mu^2 - 1)\mu}{(a_1 + a_2\mu)(\mu + q\cos\theta)} a. \tag{67}$$

Hence for the law of darkening in the general integrated radiation we have

$$I(\theta) = I_1(\theta) + I_2(\theta)$$

or

$$I(\theta) = \tfrac{3}{4}\mathfrak{F} \cos\theta + \frac{3\mu}{a_1\mu + a_2} c - \frac{3a(\mu^2 - 1)(\mu + (a_1 + a_2\mu) q \cos\theta)}{(a_1 + a_2\mu)(1 + q\cos\theta)(\mu + q\cos\theta)}. \tag{68}$$

Substituting in the above our expressions for a and c we have finally

$$I(\theta) = \mathfrak{F}\left\{\tfrac{3}{4}\cos\theta + \frac{3(a_1\mu^2 + a_2) + 2q(a_1\mu + a_2)}{2(a_1\mu + a_2)[3(a_1\mu + a_2) + 2q]} - \frac{3a_1a_2(\mu^2 - 1)(\mu - 1)[\mu + q(a_1 + a_2\mu)\cos\theta]}{2(a_1\mu + a_2)(a_1 + a_2\mu)[3(a_1\mu + a_2) + 2q](1 + q\cos\theta)(\mu + q\cos\theta)}\right\}. \tag{69}$$

The above expression reduces to the standard darkening law

$$I(\theta) = \tfrac{1}{2}\mathfrak{F}(1 + \tfrac{3}{2}\cos\theta) \tag{70}$$

for the three cases (1) $a_1 = 0$, (2) $a_2 = 0$, (3) $\mu = \infty$.

We thus see that for the models under discussion (λ and μ constants) the effect of the formation of the absorption lines is to *increase* the centre-limb contrast, and the minimum contrast corresponds to a coefficient of

darkening $u = 0{\cdot}6$. However, we shall show in Section III that our present analysis can be generalised in such a way that a reduced centre limb results.

[It may be noticed here that the equations (66) and (67) permit of a determination of the integration constants a and c by what is probably a more consistent procedure than the one we have adopted in § 12. For (66) and (67) lead to definite expressions for the emergent fluxes $\pi F_1(0)$ and $\pi F_2(0)$. Thus it is found that

$$F_1(0) = \frac{3a_1\mu}{a_1\mu + a_2}\left(\tfrac{1}{6}\mathfrak{F} + c\right) - \frac{6a_1(\mu^2 - 1)(q - \log(1+q))}{(a_1 + a_2\mu)q^2}a, \tag{71}$$

$$F_2(0) = \frac{3a_2\mu}{a_1\mu + a_2}\left(\frac{1}{6\mu}\mathfrak{F} + c\right) - \frac{6a_2\mu(\mu^2 - 1)\left(q - \mu\log\left(1 + \frac{q}{\mu}\right)\right)}{(a_1 + a_2\mu)q^2}a. \tag{72}$$

Comparing the above with our expressions (43) and (44) we can determine a and c. The expressions so obtained are not so convenient to handle as our earlier ones, and, as such, for a general orientation the cruder method we have adopted is sufficient.]

16. *The Law of Darkening. The General Case.*—Let $I_1(\theta)$ and $I_2(\theta)$ have the same meanings as in § 15. Our expressions for $I_1(\theta)$ and $I_2(\theta)$ are

$$I_1(\theta) = a_1\int_0^\infty B(\tau\cos\theta)e^{-\tau}d\tau, \tag{73}$$

$$I_2(\theta) = \frac{3(1-\epsilon)(\mu-1)}{\mu}\int_0^\infty K_2\left(\frac{\tau}{\mu}\cos\theta\right)e^{-\tau}d\tau + a_2\frac{\lambda}{\mu}\int_0^\infty B\left(\frac{\tau}{\mu}\cos\theta\right)e^{-\tau}d\tau. \tag{74}$$

Using equations (40), (41) and (42) we have

$$I_1(\theta) = \frac{3a_1\mu}{a_1\mu + a_2}\left(\tfrac{1}{4}\mathfrak{F}\cos\theta + c\right) - \frac{3a_1(\lambda\mu - 1)a}{(a_1 + a_2\lambda)(1 + q\cos\theta)}, \tag{75}$$

$$I_2(\theta) = \frac{3a_2\mu}{a_1\mu + a_2}\left(\frac{1}{4\mu}\mathfrak{F}\cos\theta + c\right) - 3\left[(1-\epsilon)(\mu-1) + a_2\cdot\frac{\lambda(\lambda\mu - 1)}{\mu(a_1 + a_2\lambda)}\right]\frac{\mu}{\mu + q\cos\theta}a. \tag{76}$$

For the case $\lambda = \mu$, $\epsilon = 1$, (75) and (76) reduce to our earlier expressions (66) and (67). When, however, the absorption lines arise as a pure scattering phenomenon, then $\epsilon = 0$ and $\lambda = 1$ and we have

$$I_1(\theta) = \frac{3a_1\mu}{a_1\mu + a_2}\left(\tfrac{1}{4}\mathfrak{F}\cos\theta + c\right) - \frac{3a_1(\mu-1)a}{1 + q\cos\theta}, \tag{77}$$

$$I_2(\theta) = \frac{3a_2\mu}{a_1\mu + a_2}\left(\frac{1}{4\mu}\mathfrak{F}\cos\theta + c\right) - \frac{3a_2(\mu-1)(\mu + a_2)a}{\mu + q\cos\theta}. \tag{78}$$

17. *The Intensity Distribution in the Continuous Spectrum.*—To determine the intensity of the emergent radiation in a wave-length λ in the continuous spectrum at the centre of the disc we have to evaluate the integral

$$I(\lambda) = \int_0^\infty B(\lambda, T_\tau)e^{-\tau}d\tau, \tag{79}$$

where T_τ is the temperature at optical depth τ and $B(\lambda, T)$ the intensity of the black-body radiation in wave-length λ for a temperature T. The variation of T with τ is, of course, given by our solution for B. The rigorous treatment of the integral (79) is too tedious, and is perhaps not worth while considering the very crude nature of the model on which we are working. A quicker method is to adopt the following procedure first used by Eddington : * "Divide the range $e^{-\tau}$ into a large number of equal parts. Calculate the temperatures T_1, T_2, T_3, . . . at the middle of each part. Take a cu. cm. of equilibrium radiation at each of these temperatures ; the simple mean gives the constitution of the emergent radiation." The above method was adopted to calculate the colour temperatures for certain special examples worked out in § 21. However, the above rough method of treating the integral was adopted only to get a general orientation to the consequences of the blanketing effect of the reversing layer. For more accurate information analytical methods should be adopted, and the equations of transfer themselves will have to be solved with greater accuracy.

We pass on now to certain numerical applications of the formulæ we have derived.

Section II

18. *The Absorption Lines formed in Pure Local Thermodynamic Equilibrium.* —To make concrete applications we need to know the value of the parameters a_1 and $\lambda(=\mu)$.

Example 1.—As a first example we shall consider the following case :—

$$a_1 = 0{\cdot}8, \quad a_2 = 0{\cdot}2, \quad \lambda = 5.$$

One finds that

$$a = 0{\cdot}003118, \quad c = 0{\cdot}1505, \quad q = 5{\cdot}916.$$

The numerical solution for B is found to be

$$B = \mathfrak{F}(0{\cdot}5374 + 0{\cdot}8929\tau - 0{\cdot}1247e^{-5{\cdot}916\tau}).$$

The relation between the boundary and effective temperatures is found to be for this case :

$$T_0^4 = 0{\cdot}4127T_e^4 \quad \text{or} \quad T_0 = 0{\cdot}8015T_e.$$

Values of B for different values of τ are tabulated in Table I (2 – column) (see also fig. 1). The expression for the fluxes are

$$F_1 = \mathfrak{F}(0{\cdot}9524 - 0{\cdot}07380e^{-5{\cdot}916\tau}),$$
$$F_2 = \mathfrak{F}(0{\cdot}0476 + 0{\cdot}07380e^{-5{\cdot}916\tau}).$$

The ratio of the emergent fluxes in the continuous spectrum and in the absorption lines is given by

$$\frac{F_1(0)}{F_2(0)} = 7{\cdot}24.$$

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* A. S. Eddington, *Internal Constitution of the Stars* (Cambridge), p. 326.

For the law of darkening we have

$$I_1(\theta) = I_1(0)\left(0{\cdot}3805 + 0{\cdot}6322\cos\theta - \frac{0{\cdot}0883}{1 + 5{\cdot}9161\cos\theta}\right),$$

$$I(\theta) = \mathfrak{F}\left(0{\cdot}5374 + 0{\cdot}75\cos\theta - \frac{0{\cdot}1247(1 + 2{\cdot}130\cos\theta)}{(1 + 5{\cdot}916\cos\theta)(1 + 2{\cdot}958\cos\theta)}\right).$$

Calculations based on the above formulæ are tabulated in Table II. One notices that the centre-limb contrast is greater than for the standard darkening.

Example 2.—As a second example we shall consider the case

$$a_1 = 0{\cdot}8, \quad a_2 = 0{\cdot}2, \quad \lambda = 10.$$

The numerical solutions are

$$B = \mathfrak{F}(0{\cdot}5547 + 0{\cdot}91467\tau - 0{\cdot}2149e^{-9{\cdot}3737\tau}),$$
$$F_1 = \mathfrak{F}(0{\cdot}9756 - 0{\cdot}07595e^{-9{\cdot}3737\tau}),$$
$$F_2 = \mathfrak{F}(0{\cdot}0244 + 0{\cdot}07595e^{-9{\cdot}3737\tau}),$$

$$B(0) = 0{\cdot}3398\mathfrak{F}\,; \quad \frac{F_1(0)}{F_2(0)} = 8{\cdot}97,$$

$$I_1(\theta) = I_1(0)\left(0{\cdot}3829 + 0{\cdot}6314\cos\theta - \frac{0{\cdot}1483}{1 + 9{\cdot}373\cos\theta}\right),$$

$$I(\theta) = \mathfrak{F}\left(0{\cdot}5547 + 0{\cdot}75\cos\theta - \frac{0{\cdot}2149(1 + 2{\cdot}624\cos\theta)}{(1 + 9{\cdot}373\cos\theta)(1 + 0{\cdot}937\cos\theta)}\right).$$

19. *Absorption Lines formed as Pure Scattering Phenomenon.*—To see the effect of the "ϵ" factor on the law of darkening the extreme case $\lambda = 1$ was considered for the examples considered in § 18.

Example 3.—

$$a_1 = 0{\cdot}8, \qquad a_2 = 0{\cdot}2, \qquad \lambda = 1, \qquad \mu = 5.$$

The results of the calculations are

$$B = \mathfrak{F}(0{\cdot}5464 + 0{\cdot}89297\tau - 0{\cdot}0464e^{-3{\cdot}557\tau}),$$
$$F_1 = \mathfrak{F}(0{\cdot}9524 - 0{\cdot}0549e^{-3{\cdot}557\tau}),$$
$$F_2 = \mathfrak{F}(0{\cdot}0476 + 0{\cdot}0549e^{-3{\cdot}557\tau}),$$

$$B(0) = 0{\cdot}5\mathfrak{F}\,; \quad \frac{F_1(0)}{F_2(0)} = 8{\cdot}75,$$

$$I_1(\theta) = I_1(0)\left(0{\cdot}3824 + 0{\cdot}6248\cos\theta - \frac{0{\cdot}03249}{1 + 3{\cdot}55\cos\theta}\right).$$

Example 4.—

$$a_1 = 0{\cdot}8, \qquad a_2 = 0{\cdot}2, \qquad \lambda = 1, \qquad \mu = 10.$$

$$B = \mathfrak{F}(0{\cdot}5687 + 0{\cdot}91467\tau - 0{\cdot}0687e^{-4{\cdot}967\tau}),$$
$$F_1 = \mathfrak{F}(0{\cdot}9756 - 0{\cdot}0505e^{-4{\cdot}967\tau}),$$
$$F_2 = \mathfrak{F}(0{\cdot}0244 + 0{\cdot}0505e^{-4{\cdot}967\tau}),$$

$$B(0) = 0{\cdot}5\mathfrak{F}\,; \quad \frac{F_1(0)}{F_2(0)} = 12{\cdot}4,$$

$$I_1(\theta) = I(0)\left(0{\cdot}3864 + 0{\cdot}6214\cos\theta - \frac{0{\cdot}0467}{1 + 4{\cdot}96\cos\theta}\right).$$

20. *The Discussion of the Results.*—The values of B for different values of τ for the four examples considered in §§ 18 and 19 are tabulated in Table I and the variations are graphically illustrated in fig. 1. The results for darkening are contained in Tables II and III.

An examination of Table I and fig. 1 shows that for a fixed a_1 (and a_2) as $\mu \to \infty$ the appropriate limiting solution is

$$B = \frac{1}{2a_1}\mathfrak{F}(1 + \tfrac{3}{2}\tau), \tag{80}$$

Table I

$B(\tau)$ in Units of $\mathfrak{F}$ ($a_1 = 0{\cdot}8$, $a_2 = 0{\cdot}2$).

τ	$\lambda = \mu = 5$	$\lambda = \mu = 10$	$\lambda = 1$; $\mu = 5$	$\lambda = 1$; $\mu = 5$
0	0·4127	0·3398	0·5	0·5
·02	0·4445	0·3949	0·5210	0·5248
·05	0·4893	0·4660	0·5522	0·5608
·1	0·5577	0·5620	0·6032	0·6183
·2	0·6778	0·7040	0·7022	0·7261
·3	0·7841	0·8162	0·7983	0·8276
·4	0·8829	0·9155	0·8924	0·9251
·5	0·9774	1·0100	0·9850	1·0203
·6	1·0696	1·1027	1·0766	1·1140
·8	1·2506	1·2863	1·2580	1·2991
1·0	1·4299	1·4693	1·4380	1·4828
1·2	1·609	1·652	1·617	1·666
1·4	1·787	1·835	1·796	1·849
1·6	1·966	2·018	1·975	2·032
1·8	2·145	2·201	2·153	2·215
2·0	2·323	2·384	2·332	2·398
3·0	3·216	3·299	3·225	3·313
4·0	4·109	4·213	4·118	4·227

and a certain interval on the B-axis. In the case of the absorption lines arising as a pure scattering phenomenon this interval is

$$< \frac{1}{2a_1}\mathfrak{F}, \quad \tfrac{1}{2}\mathfrak{F} >, \tag{81}$$

and in the case of pure local thermodynamic equilibrium the interval is

$$< \frac{1}{2a_1}\mathfrak{F}, \quad 0 >. \tag{82}$$

The apparent contradiction between (80), which predicts a boundary temperature T_0 related to the effective temperature T_e by the relation

$$T_0^4 = \frac{1}{2a_1}T_e^4, \tag{83}$$

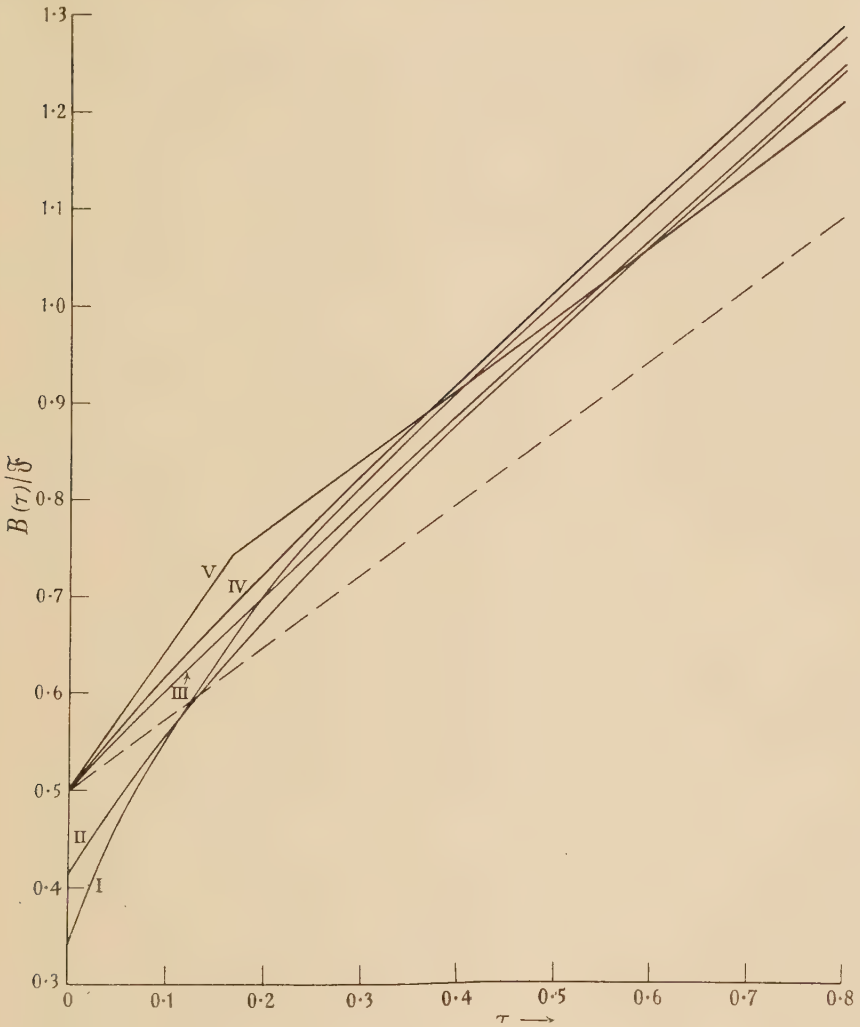

FIG. 1.—*Variation of $B(\tau)$ with τ including the blanketing effect of the reversing layer.*

I $\lambda=\mu=10$, $a_2=0{\cdot}2$.
II $\lambda=\mu=5$, $a_2=0{\cdot}2$.
III $\lambda=1$, $\mu=5$, $a_2=0{\cdot}2$.
IV $\lambda=1$, $\mu=10$, $a_2=0{\cdot}2$.
V $\lambda=1$, $\mu=10$, $\tau<1/6$; $\lambda=\mu=1$, $\tau\geqslant 1/6$; $a_1=0{\cdot}8$, $a_2=0{\cdot}2$.
The - - - Curve is the "standard law" : $B(\tau)=\mathfrak{F}(1+\frac{3}{2}\tau)$.

and our earlier results in § 14 (where we saw that as $\mu\to\infty$ the boundary temperature is zero for the case $\lambda=\mu$, and $T_0=(2)^{-4}T_e$ for the case $\lambda=1$) is now removed.

Fig. 1 also explains why the blanketing effect of the reversing layer is to produce a larger centre-limb contrast than the standard darkening, $u=0{\cdot}6$. (It may be pointed out explicitly here that the law of darkening *is* the standard one both for $\mu=1$ and $\mu=\infty$.) The reason is, in the immediate

TABLE II

Law of Darkening in Pure Local Thermodynamic Equilibrium

Cos θ	$a_1=0{\cdot}8$; $a_2=0{\cdot}2$; $\lambda=\mu=5$		$a_1=0{\cdot}8$; $a_2=0{\cdot}2$; $\lambda=\mu=10$		Standard darkening $u=3/5$
	$\frac{I_1(\theta)}{I_1(0)}$	$\frac{I_1(\theta)+I_2(\theta)}{I_1(0)+I_2(0)}$	$\frac{I_1(\theta)}{I_1(0)}$	$\frac{I_1(\theta)+I_2(\theta)}{I_1(0)+I_2(0)}$	
1	1	1	1	1	1
0·9	0·9356	0·9400	0·9354	0·9386	0·94
0·8	0·8709	0·8797	0·8706	0·8768	0·88
0·6	0·7405	0·7578	0·7394	0·7514	0·76
0·4	0·6072	0·6330	0·6042	0·6163	0·64
0·2	0·4665	0·4997	0·4576	0·4808	0·52
0·1	0·3883	0·4234	0·3695	0·3963	0·44
0	0·2922	0·3241	0·2346	0·2684	0·40

TABLE III

Law of Darkening (the absorption lines formed as a pure scattering phenomenon)
($a_1=0{\cdot}8$; $a_2=0{\cdot}2$; $\lambda=1$)

Cos θ	$I_1(\theta)/I_1(0)$ ($\mu=5$)	$I_1(\theta)/I_1(0)$ ($\mu=10$)
1	1	1
0·9	0·9369	0·9372
0·8	0·8737	0·8742
0·6	0·7468	0·7475
0·4	0·6188	0·6193
0·2	0·4883	0·4873
0·1	0·4209	0·4173
0	0·3499	0·3397

outer layers the deviations from a linear increase of B with τ become pronounced, and it is precisely these regions which contribute most to the emergent radiation near the limb. One further notices that for equal values of a_1, a_2 and μ the photosphere is maintained uniformly at a higher temperature in the case where the absorption lines arise as a purely scattering phenomenon than in the case of pure local thermodynamical equilibrium.

21. *Applications to the Colour Temperature of the Sun.*—Assuming Abbot's value for the effective temperature of the Sun, namely, $T_e=5740°$, the mean intensities of the emergent radiation (at the centre of the disc) in the wave-lengths 4157 A., 6235 A. and 12470 A. for the four examples considered in §§ 18 and 19 were determined by the method sketched in § 17. The results of the calculation are summarised in Table IV. The corresponding

TABLE IV

The Effect of the Reversing Layer on the Colour Temperature

Description	Mean Intensity			$\frac{\text{Intensity}_{6235}}{\text{Intensity}_{4157}}$	Colour Temperature
	4157 A.	6235 A.	12470 A.		
Pure local thermodynamic equilibrium :					
(1) $\lambda=\mu=5$, $a_2=0{\cdot}2$	0·00408	0·02463	0·1783	6·04	6475°
(2) $\lambda=\mu=10$, $a_2=0{\cdot}2$	0·00422	0·02522	0·1807	5·98	6500°
Absorption lines formed as a pure scattering phenomenon :					
(1) $\lambda=1$, $\mu=5$, $a_2=0{\cdot}2$	0·00413	0·02496	0·1801	6·04	6475°
(2) $\lambda=1$, $\mu=10$, $a_2=0{\cdot}2$	0·00429	0·02564	0·1830	5·97	6510°
No Blanketing Effect ($a_2=0$)	0·00344	0·0221	0·168	6·43	6220°

colour temperatures (defined by the ratios of the intensities at the wave-lengths 4157 A. and 6235 A.) are given in the last column. One sees that the blanketing effect, as we have considered so far, is almost sufficient to account for Plaskett's high value for the true colour temperature of the Sun. The results in Table IV show that the colour temperature is not particularly sensitive to the "model" or to the particular values of λ and μ, provided μ is greater than about 5. It is the value of a_2 that is important. The particular value $a_2=0{\cdot}2$ adopted in our calculations is not unreasonable, especially when it is realised that a value of $\eta=1/9$ in the Milne-Lindblad treatment predicts a boundary temperature identical with that predicted by (82)—(case $\mu=\infty$)—for $a_1=0{\cdot}8$, $a_2=0{\cdot}2$.

Section III

22. In Section II we saw that on the models there discussed the theory predicts a law of darkening with a higher centre-limb contrast than the standard darkening, thus making the general agreement between the theory and the observations worse in this respect. But on the Milne-Lindblad treatment the centre-limb contrast is less than that on the standard darkening, and this suggests how we should generalise our earlier analysis in order that the theory may predict a law of darkening more in agreement with the observations.

Now in an actual stellar atmosphere both $\xi(=\alpha/\kappa)$ and ϵ will not be constants. It is, of course, clear that in the deeper layers ϵ will tend to unity (pure local thermodynamical equilibrium), while in the reversing layer ϵ will be more near zero. Further, the ratio α/κ will decrease inwards

(κ predominates in the deeper layers). Thus a better approximation to the outer layers of a star is obtained by dividing the atmosphere into two parts, ξ and ϵ taking different values in the outer and the deeper layers respectively, but constant in each of the regions separately. The analysis required for this more general problem does not present any new difficulty.

23. Let ξ and ϵ take the values ξ_1, ϵ_1 in the outer layers and the values ξ_2, ϵ_2 in the deeper layers. Let the surface of demarcation between the (ξ_1, ϵ_1) and the (ξ_2, ϵ_2) regions occur at $\tau = \tau_0$.* As before we shall now define the quantities λ_1, λ_2, μ_1, μ_2, as follows :—

$$\left.\begin{aligned} \lambda_1 &= 1 + \epsilon_1\xi_1\,; \quad \mu_1 = 1 + \xi_1, \\ \lambda_2 &= 1 + \epsilon_2\xi_2\,; \quad \mu_2 = 1 + \xi_2. \end{aligned}\right\} \tag{84}$$

Further, let q_1 and q_2 be defined in terms of (λ_1, μ_1) and (λ_2, μ_2) respectively exactly as in equation (38). Then by an analysis similar to that in §§ 9 and 10 we find that for $\tau \leqslant \tau_0$

$$\left.\begin{aligned} K_1 &= ae^{-q_1\tau} + be^{q_1\tau} + \frac{a_1\mu_1}{a_1\mu_1 + a_2}\left(\tfrac{1}{4}\mathfrak{F}\tau + c_1\right), \\ K_2 &= -\mu_1 ae^{-q_1\tau} - \mu_1 be^{q_1\tau} + \frac{a_2\mu_1}{a_1\mu_1 + a_2}\left(\tfrac{1}{4}\mathfrak{F}\tau + c_1\right), \\ F_1 &= \frac{a_1\mu_1}{a_1\mu_1 + a_2}\mathfrak{F} - 4aq_1e^{-q_1\tau} + 4bq_1e^{q_1\tau}, \\ F_2 &= \frac{a_2}{a_1\mu_1 + a_2}\mathfrak{F} + 4aq_1e^{-q_1\tau} - 4bq_1e^{q_1\tau}, \\ B &= \frac{3}{a_1 + a_2\lambda_1}(K_1 + \lambda_1K_2). \end{aligned}\right\} \tag{85}$$

In the above equations a, b and c_1 are three integration constants. Similarly for $\tau \geqslant \tau_0$ we have

$$\left.\begin{aligned} K_1 &= Ae^{-q_2\tau} + \frac{a_1\mu_2}{a_1\mu_2 + a_2}\left(\tfrac{1}{4}\mathfrak{F}\tau + c_2\right), \\ K_2 &= -\mu_2 Ae^{-q_2\tau} + \frac{a_2\mu_2}{a_1\mu_2 + a_2}\left(\tfrac{1}{4}\mathfrak{F}\tau + c_2\right), \\ F_1 &= \frac{a_1\mu_2}{a_1\mu_2 + a_2}\mathfrak{F} - 4Aq_2e^{-q_2\tau}, \\ F_2 &= \frac{a_2}{a_1\mu_2 + a_2}\mathfrak{F} + 4Aq_2e^{-q_2\tau}, \\ B &= \frac{3}{a_1 + a_2\lambda_2}(K_1 + \lambda_2K_2), \end{aligned}\right\} \tag{86}$$

where A and c_2 are two other integration constants.

At $\tau = 0$ we set as usual

$$6K_1(0) = F_1(0)\,; \qquad 6K_2(0) = F_2(0). \tag{87}$$

* We shall show later that the continuity of temperature at $\tau = \tau_0$ requires that $\xi_1 \cdot \epsilon_1 = \xi_2 \cdot \epsilon_2$ (*cf.* equation (88)).

Further, at $\tau = \tau_0$ the two sets of solutions must give the same values for K_1, K_2, F_1, F_2 and B. The continuity of K_1, K_2, F_1 and F_2 at $\tau = \tau_0$ provide *three* further conditions,* which together with the two equations provided by (87) will determine the five constants a, b, c_1, c_2 and A. On the other hand, the continuity of B requires that $\lambda_1 = \lambda_2$ or

$$\epsilon_1 \xi_1 = \epsilon_2 \xi_2. \tag{88}$$

(88) can be regarded as a statement of Kirchoff's law for our problem.

The equations which determine the integration constants can be reduced to the following set :—

$$(6+4q_1)a+(6-4q_1)b+\frac{6a_1\mu_1}{a_1\mu_1+a_2}c_1-\frac{a_1\mu_1}{a_1\mu_1+a_2}\mathfrak{F}=0,$$

$$(\mu_1-1)a+(\mu_1-1)b-\frac{\mu_1}{a_1\mu_1+a_2}c_1+\tfrac{1}{6}\mathfrak{F}=0,$$

$$(\tfrac{1}{4}\mathfrak{F}\tau_0+c_1)+e^{-q_2\tau_0}\left(\frac{\mu_2}{\mu_1}-1\right)A-\frac{\mu_2}{a_1\mu_2+a_2}\left(\frac{a_2}{\mu_1}+a_1\right)(\tfrac{1}{4}\mathfrak{F}\tau_0+c_2)=0,$$

$$\left(\frac{\mu_1}{\mu_2}-1\right)e^{-q_1\tau_0}a+\left(\frac{\mu_1}{\mu_2}-1\right)e^{q_1\tau_0}b-\frac{\mu_1}{a_1\mu_1+a_2}\left(\frac{a_2}{\mu_2}+a_1\right)(\tfrac{1}{4}\mathfrak{F}\tau_0+c_1)+(\tfrac{1}{4}\mathfrak{F}\tau_0+c_2)=0,$$

$$4q_1e^{-q_1\tau_0}a-4q_1e^{q_1\tau_0}b+\frac{a_2}{a_1\mu_1+a_2}\mathfrak{F}-4q_2e^{-q_2\tau_0}A-\frac{a_2}{a_1\mu_2+a_2}\mathfrak{F}=0.$$

The explicit solutions of the above equations are too complicated to be of any interest. It is more convenient to solve them numerically for each special case.

24. To illustrate the use of the formulæ of § 23 we shall consider the following extreme case :—

$$\left.\begin{array}{llll} \epsilon_1=0, & \xi_1=9, & \lambda_1=1, \quad \mu_1=10, & (\tau \leqslant \tau_0) \\ \epsilon_2=1, & \xi_2=0, & \lambda_2=\mu_2=1, & (\tau \geqslant \tau_0) \end{array}\right\} \tag{89}$$

—*i.e.* we are considering the case of a purely scattering atmosphere placed above material in pure local thermodynamic equilibrium and no scattering. (This case therefore corresponds to the model which underlies the cruder Milne-Lindblad treatment.) The general results are best seen from a numerical example.

Taking

$$a_1=0{\cdot}8, \qquad a_2=0{\cdot}2, \qquad \tau_0=1/6, \tag{90}$$

the numerical solutions are found to be :

$$\left.\begin{array}{l} K_1=\mathfrak{F}(0{\cdot}00310e^{-4{\cdot}967\tau}-0{\cdot}00129e^{4{\cdot}967\tau}+0{\cdot}24390\tau+0{\cdot}14631), \\ K_2=\mathfrak{F}(-0{\cdot}03095e^{-4{\cdot}967\tau}+0{\cdot}01288e^{4{\cdot}967\tau}+0{\cdot}06095\tau+0{\cdot}03658), \\ F_1=\mathfrak{F}(0{\cdot}9756-0{\cdot}0614e^{-4{\cdot}967\tau}-0{\cdot}0256e^{4{\cdot}967\tau}), \\ F_2=\mathfrak{F}(0{\cdot}0244+0{\cdot}0614e^{-4{\cdot}967\tau}+0{\cdot}0256e^{4{\cdot}967\tau}), \\ B=\mathfrak{F}(0{\cdot}5487+0{\cdot}9146\tau-0{\cdot}0836e^{-4{\cdot}967\tau}+0{\cdot}0348e^{4{\cdot}967\tau}). \end{array}\right\} (\tau \leqslant 1/6)$$

* *Three* boundary conditions and not four, since $F_1+F_2=\mathfrak{F}$ in the two regions separately.

$$\left.\begin{aligned} K_1 &= \mathfrak{F}(-0{\cdot}0174e^{-\sqrt{3}\tau} + 0{\cdot}2\tau + 0{\cdot}16508), \\ K_2 &= \mathfrak{F}(+0{\cdot}0174e^{-\sqrt{3}\tau} + 0{\cdot}05\tau + 0{\cdot}04127), \\ F_1 &= \mathfrak{F}(0{\cdot}8 + 0{\cdot}1206e^{-\sqrt{3}\tau}), \\ F_2 &= \mathfrak{F}(0{\cdot}2 - 0{\cdot}1206e^{-\sqrt{3}\tau}), \\ B &= \mathfrak{F}(0{\cdot}75\tau + 0{\cdot}61905). \end{aligned}\right\} (\tau \geqslant 1/6)$$

At $\tau = 0$

$$F_1(0) = 0{\cdot}8887\mathfrak{F}\,; \qquad F_2(0) = 0{\cdot}1113\mathfrak{F}\,; \qquad B(0) = \tfrac{1}{2}\mathfrak{F}.$$

At $\tau = \tau_0$ the two sets of formulæ both give the following values for the different quantities expressed in units of $\mathfrak{F}$:—

$$\begin{aligned} F_1(\tau_0) &= 0{\cdot}8903\,; \qquad K_1(\tau_0) = 0{\cdot}1854\,; \qquad B(\tau_0) = 0{\cdot}7440, \\ F_2(\tau_0) &= 0{\cdot}1097\,; \qquad K_2(\tau_0) = 0{\cdot}0626. \end{aligned}$$

In fig. 1 the variation of B with τ is illustrated. To find the law of darkening we have to calculate $I_1(\theta)$. In integrating $B(\tau \cos\theta)e^{-\tau}$ we have to use the different expressions for B in the different regions. The final expression is

$$I_1(\theta) = \mathfrak{F}\left\{0{\cdot}6082 + 0{\cdot}7521\cos\theta + \frac{0{\cdot}0348(e^{(4{\cdot}96\cos\theta - 1)/6} - 1)}{4{\cdot}96\cos\theta - 1} - \frac{0{\cdot}0836(1 - e^{-(4{\cdot}96\cos\theta + 1)/6})}{4{\cdot}96\cos\theta + 1}\right\}. \quad (91)$$

Calculations based on the above formula for darkening are given in Table V and compared with a linear law with a coefficient of darkening $u = 0{\cdot}56$,

Table V

Cos θ	$\frac{I(\theta)}{I(0)}$ according to (91)	$\frac{I(\theta)}{I(0)}$ for $u = {\cdot}56$
1·0	1	1
0·9	0·9443	0·944
0·8	·8884	·888
0·6	·7768	·776
0·4	·6652	·664
0·2	·5534	·552
0·1	·4977	·496
0	·4418	·440

which value of u, according to Mulders, fits the solar observations best. By an accidental coincidence our numerical example reproduces the observed law remarkably well.

However, the prediction concerning the colour temperature is not so satisfactory. The colour temperature to be expected on the basis of our present model was also estimated by the method sketched in § 17. It is

found that the resulting colour temperature is 6350°, which is rather low. The details of the calculation are given below :

Wave-length	Intensity
4137 A.	0·00386
6235 A.	0·02408
12470 A.	0·1774
Intensity at 6235 A. / Intensity at 4137 A.	6·24
T colour	6350°

The reason why our present colour temperature is much lower is of course obvious. The temperature gradient in the deeper layers is now less than on the models considered in Section II. It would, however, be possible by a more suitable adjustment of the constants λ_1, μ_1, λ_2 and μ_2 to predict a higher colour temperature than 6350° without at the same time making too great a departure from the law of darkening we have found in the above numerical example.

25. From the numerical example of § 24 it is now clear that the departures from the standard law of darkening that have been observed in the case of the Sun can be accounted for on our present analysis by considering two regions with different values for λ and μ. Clearly the formal analysis can be extended for more regions. We have solutions of the type (85) for each intermediate region, and finally a system of the type (86) for the deepest region. At each surface of demarcation there will be three boundary conditions to be satisfied, and the total number of boundary conditions provided will be just sufficient to make the problem determinate. Perhaps the best representation of the physical situation will be by a division of the outer layers into three separate regions with $\epsilon = 0{\cdot}1$ for the outermost region and $\epsilon = 1/2$ and $\epsilon = 1$ for the intermediate and the deepest regions respectively. But the calculations presented here are sufficient to illustrate the general utility of considering blanketing in the way we have done in this paper.

Summary.—In this paper an attempt has been made to include in a theory of the radiative equilibrium of the outer layers of a star the fact of the formation of the absorption lines in the reversing layer immediately above the conventional photosphere. It has been found possible to treat the problem fairly rigorously by introducing the absorption lines *uniformly* in the whole spectral range. The conclusion is reached that Plaskett's observed high value for the true colour temperature of the Sun can be accounted for. Further, it is shown that departures of the observed law of darkening from the standard darkening (with a coefficient $u = 0{\cdot}6$) can also be explained by considering a "scattering" atmosphere placed above a region of local thermodynamical equilibrium.

In conclusion, it is a pleasure to record here my best thanks to Professor H. H. Plaskett for his encouragement and advice during the progress of the work.

Trinity College, Cambridge :
1935 *November* 1.

Note added on November 29.—Mr. Greaves has kindly pointed out to the writer that the rather crude method adopted in this paper of estimating the colour temperatures is likely to lead to underestimates. Thus, in the case $a_2 = 0$ (no blanketing effect) the crude method of evaluating the integrals gives a colour temperature of 6220°, while Milne's accurate method * gave the higher value 6400°. In the same way, an accurate evaluation of the colour temperatures for the models considered in this paper might lead to colour temperatures more nearly 6700°. This higher value would definitely be in better agreement with observations. In this connection Mr. Greaves especially refers to Shajn's recent work.†

THE STABILITY OF GASEOUS STARS.

(Second Paper.)

T. G. Cowling, M.A., D.Phil.

1. In an earlier paper ‡ the problem of the stability of gaseous stars was considered. It was shown that when convection is not present in a star, so that the transfer of heat is by radiation, the star is vibrationally stable save when the ratio γ of specific heats of the material is nearly equal to the critical value 4/3. In this paper the vibrations of a star in which convection is present are considered. We find that there is stability unless γ is nearly equal to 4/3, or the rate of generation of heat energy increases extremely rapidly with the temperature. For example, it is estimated that if the rate of generation varies as the twentieth power of the temperature there is stability unless γ is less than 1·44.

This differs considerably from the earlier results of Jeans § and Eddington,|| who both suggested that stability will certainly only exist if the rate of generation varies more slowly than the cube of the temperature. Jeans assumed that δr, the radial displacement of matter whose equilibrium distance from the centre of the star is r, is proportional to r at any instant of an oscillation ; actually $\delta r/r$ increases steadily with r, and the effect of this increase on stability is large. In Eddington's work an attempt was made to take the increase into account, but its effect was considerably underestimated. Convection is not the chief cause of the difference ; it

* *Handbuch der Astrophysik*, 3/1, 153, 1930.
† *M.N.*, **94**, 7, 652, 1934.
‡ *M.N.*, **94**, 768, 1934. This paper will be referred to as I.
§ *M.N.*, **85**, 914, 1925 ; **87**, 400, 720, 1927.
|| *Internal Constitution of the Stars*, p. 201.

does not greatly affect stability, save through the equilibrium density gradient which is associated with its presence.

The importance of the result is evident. It has been a real difficulty, usually glossed over, that theory appeared to predict that gaseous stars can only be stable if the law of variation of generation of energy with temperature lies within certain narrow limits, and that there is no reason why it should do so. This difficulty is now removed, and the case for a belief that the majority of stars are gaseous throughout is correspondingly strengthened. It must be emphasised, however, that if the rate of generation does increase rapidly with the temperature, the transfer of energy in the interior will be by convection, not radiation.

2. Meteorologists have developed methods for use in problems involving convection.* The velocity c of the gas at any point, whose components are u, v, w, is divided into a mass velocity c_0 with components u_0, v_0, w_0, and a turbulent velocity C with components U, V, W. Mean values taken on averaging over a sufficiently large mass of gas for the local irregularities due to the turbulent motion to be removed are represented by a bar over the symbol. Let ρ be the density ; then c_0 is defined by the equations

$$\overline{\rho u} = \bar{\rho} u_0, \qquad \overline{\rho v} = \bar{\rho} v_0, \qquad \overline{\rho w} = \bar{\rho} w_0, \tag{1}$$

so that

$$\overline{\rho U} = \overline{\rho V} = \overline{\rho W} = 0.$$

It is to be observed that the mass velocity is in general not equal to the mean velocity whose components are $\bar{u}$, $\bar{v}$, $\bar{w}$.

It is supposed that masses of gas are continually becoming detached from the surrounding matter, moving bodily through a certain distance, and then being reabsorbed into the main mass of material. This enables us to find approximate expressions for certain mean values appearing in the macroscopic equations. To illustrate the method we proceed to evaluate $\bar{U}$. In so doing we assume that the pressure p is not affected appreciably by the turbulence, and that $\bar{\rho}$, p are functions of x alone.

Consider the value of $\bar{U}$ when $x = 0$. If S is a large area of the plane $x = 0$, then

$$S\bar{U} = \iint U dS,$$

where the integration is extended over the area S. Let U be positive over a part S_1 of S and negative over the remaining part $S - S_1$; let its mean values over these parts be U_1, U_2. Then

$$S\bar{U} = S_1 U_1 + (S - S_1) U_2.$$

Again,

$$\iint \rho U dS = S\overline{\rho U} = 0 ;$$

or if ρ_1, ρ_2 are suitable mean values of ρ over the parts S_1, $S - S_1$ of S,

$$\rho_1 U_1 S_1 + \rho_2 U_2 (S - S_1) = 0.$$

* For an account of these methods see, *e.g.*, Brunt's *Physical and Dynamical Meteorology*, chaps. xii, xiii and xv.

Hence $$S\bar{U} = U_1 S_1 (1 - \rho_1/\rho_2).$$

Now ρ_1 differs from the value of $\bar{\rho}$ on S, since gas for which $U > 0$ has moved a certain distance through the mass in the positive direction of Ox, expanding adiabatically as it moves.* Hence ρ_1 will be given by

$$\rho_1/\rho' = (p/p')^{1/\gamma},$$

where p is the pressure at $x = 0$, and p', ρ' are the pressure and mean density at a point for which x is negative, say $x = -l$. If l is small compared with the scale of variation of the macroscopic properties of the gas,

$$p' = p - l\frac{\partial p}{\partial x}, \qquad \rho' = \bar{\rho} - l\frac{\partial \bar{\rho}}{\partial x},$$

the quantities on the right of these equations being evaluated at $x = 0$. Hence, to the same approximation,

$$\rho_1/\bar{\rho} = 1 + \frac{l}{\gamma p}\frac{\partial p}{\partial x} - \frac{l}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial x}.$$

Similarly it may be shown that approximately

$$\rho_2/\bar{\rho} = 1 - \frac{l}{\gamma p}\frac{\partial p}{\partial x} + \frac{l}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial x}.$$

Thus

$$S\bar{U} = 2 S_1 U_1 l\left(\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial x} - \frac{1}{\gamma p}\frac{\partial p}{\partial x}\right).$$

Now S_1 is approximately half of S; also U_1 is a mean velocity of turbulence, which we denote by q. Hence finally

$$\begin{aligned}\bar{U} &= lq\left(\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial x} - \frac{1}{\gamma p}\frac{\partial p}{\partial x}\right) \\ &= -lq\left(\frac{1}{\bar{T}}\frac{\partial \bar{T}}{\partial x} - \frac{(\gamma - 1)}{\gamma p}\frac{\partial p}{\partial x}\right).\end{aligned} \tag{2}$$

This expression is valid even if p, $\bar{\rho}$ depend on other variables beside x.

By the same method it may be shown that, if E is the energy of the material per unit mass, the rate of convection of heat parallel to Ox is †

* Actually the variation of temperature is not adiabatic in the outer layers of a star, as radiation dominates convection there. Also it must be observed that there is no true ratio γ of specific heats for a mixture of ionised gas and radiation. It is, however, convenient to assume that in an adiabatic change p, ρ and the temperature T satisfy the ordinary gas relations

$$p \propto \rho^\gamma, \qquad T \propto \rho^{\gamma-1}, \qquad p \propto \rho T.$$

† The expression usually given for this quantity differs from (3) in having c_p, the specific heat at constant pressure, in place of c_v, because the absorption of a moving mass of gas into the main body of the material takes place at constant pressure. The above expression is the correct one, since the gain of actual thermal energy when the temperature increases by ΔT is $c_v \Delta T$ per unit mass; the difference between this and $c_p \Delta T$ is mechanical work done by the gas, which appears at a different place in the exact equations.

$$\overline{\rho EU} = -\bar{\rho} l q c_v \left(\frac{\partial \bar{T}}{\partial x} - \frac{\bar{T}(\gamma - 1)}{\gamma p} \frac{\partial p}{\partial x} \right) \tag{3}$$

per unit area, relative to the mass, where c_v is the specific heat per unit mass at constant volume. The argument only shows that the l and q in this expression are of the same order as those in (2); but for simplicity it is assumed that they are actually equal to them.

3. Consider now the motion of the gas. The viscous pressure-tensor has components P_{xx}, P_{xy}, P_{yx}, P_{yy}, etc., where

$$P_{xx} = -\tfrac{2}{3}\mu \left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right),$$

$$P_{xy} = P_{yx} = -\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \text{ etc.}, \tag{4}$$

μ being the coefficient of viscosity (normally of radiative viscosity). The total pressure-tensor is found on adding to this the hydrostatic pressure p_1.

The equations governing the motion are the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0; \tag{5}$$

the equations of motion, which may be written in the form

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}$$

$$= \rho X - \frac{\partial p_1}{\partial x} - \frac{\partial P_{xx}}{\partial x} - \frac{\partial P_{xy}}{\partial y} - \frac{\partial P_{xz}}{\partial z}, \text{ etc.}, \tag{6}$$

where X, Y, Z are the components of the body force on the gas per unit mass; and the equation of thermal energy, which may be expressed as

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho Eu)}{\partial x} + \frac{\partial(\rho Ev)}{\partial y} + \frac{\partial(\rho Ew)}{\partial z}$$

$$= -p_1 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - P_{xx}\frac{\partial u}{\partial x} - P_{yy}\frac{\partial v}{\partial y} - P_{zz}\frac{\partial w}{\partial z}$$

$$- P_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) - P_{yz}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) - P_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

$$+ \rho\epsilon_1 - \frac{\partial (F_1)_x}{\partial x} - \frac{\partial (F_1)_y}{\partial y} - \frac{\partial (F_1)_z}{\partial z}, \tag{7}$$

where ϵ_1 is the rate of generation of energy per unit mass, and $(F_1)_x$, $(F_1)_y$, $(F_1)_z$ are the components of the flux F_1 of thermal energy relative to the matter.* From these equations others can be obtained on taking mean values.

* This equation does not seem to appear in the standard books. It is, however, easily derived on observing that the left-hand side, multiplied by dt, represents the change during dt in the energy per unit volume of gas initially at (x, y, z); this is equated on the right to the sum of the work done on the gas by the mechanical forces, the energy generated, and the energy flowing into the gas.

On taking the mean of (5), we have

$$\frac{\partial \bar{\rho}}{\partial t}+\frac{\partial}{\partial x}(\bar{\rho}u_0)+\frac{\partial}{\partial y}(\bar{\rho}v_0)+\frac{\partial}{\partial z}(\bar{\rho}w_0)=0.$$

If we set

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y}+w_0\frac{\partial}{\partial z}, \tag{8}$$

this reduces to

$$\frac{D\bar{\rho}}{Dt}=-\bar{\rho}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right). \tag{9}$$

Again, from (6), on setting $u=u_0+U$, etc., on the left-hand side, and averaging

$$\frac{\partial}{\partial t}(\bar{\rho}u_0)+\frac{\partial}{\partial x}(\bar{\rho}u_0{}^2+\overline{\rho U^2})+\frac{\partial}{\partial y}(\bar{\rho}u_0v_0+\overline{\rho UV})+\frac{\partial}{\partial z}(\bar{\rho}u_0w_0+\overline{\rho UW})$$

$$=\bar{\rho}X-\frac{\partial \bar{p}_1}{\partial x}-\frac{\partial \bar{P}_{xx}}{\partial x}-\frac{\partial \bar{P}_{xy}}{\partial y}-\frac{\partial \bar{P}_{xz}}{\partial z},$$

whence, by (9),

$$\bar{\rho}\frac{Du_0}{Dt}=\bar{\rho}X-\frac{\partial \bar{p}_1}{\partial x}$$

$$-\frac{\partial}{\partial x}(\bar{P}_{xx}+\overline{\rho U^2})-\frac{\partial}{\partial y}(\bar{P}_{xy}+\overline{\rho UV})+\frac{\partial}{\partial z}(\bar{P}_{xz}+\overline{\rho UW}). \tag{10}$$

This expresses the well-known result that as a result of turbulence the pressure is increased by the Reynolds stresses $\overline{\rho U^2}$, $\overline{\rho UV}$, etc.

From (10) we obtain the equation of energy of the mass motion,

$$\tfrac{1}{2}\bar{\rho}\frac{D}{Dt}(c_0{}^2)=\bar{\rho}(u_0X+v_0Y+w_0Z)-u_0\frac{\partial \bar{p}_1}{\partial x}-v_0\frac{\partial \bar{p}_1}{\partial y}-w_0\frac{\partial \bar{p}_1}{\partial z}$$

$$-u_0\left\{\frac{\partial}{\partial x}(\bar{P}_{xx}+\overline{\rho U^2})+\frac{\partial}{\partial y}(\bar{P}_{xy}+\overline{\rho UV})+\frac{\partial}{\partial z}(\bar{P}_{xz}+\overline{\rho UW})\right\}$$

$$-v_0\{\ldots.\}-w_0\{\ldots.\}. \tag{11}$$

There is also an equation of turbulent energy. By (5) and (6),

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho c^2)=\rho(uX+vY+wZ)-u\frac{\partial p_1}{\partial x}-v\frac{\partial p_1}{\partial y}-w\frac{\partial p_1}{\partial z}$$

$$-u\left(\frac{\partial P_{xx}}{\partial x}+\frac{\partial P_{xy}}{\partial y}+\frac{\partial P_{xz}}{\partial z}\right)-v(\ldots.)-w(\ldots.)$$

$$-\frac{\partial}{\partial x}(\tfrac{1}{2}\rho c^2u)-\frac{\partial}{\partial y}(\tfrac{1}{2}\rho c^2v)-\frac{\partial}{\partial z}(\tfrac{1}{2}\rho c^2w).$$

Hence, on setting $u = u_0 + U$, etc., and taking mean values,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\bar{\rho}\bar{c}_0{}^2 + \tfrac{1}{2}\overline{\rho C^2}) = \bar{\rho}(u_0X + v_0Y + w_0Z) - \overline{u\frac{\partial p_1}{\partial x}} - \overline{v\frac{\partial p_1}{\partial y}} - \overline{w\frac{\partial p_1}{\partial z}}$$

$$- \overline{u\left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} + \frac{\partial P_{xz}}{\partial z}\right)} - \overline{v(\ldots)} - \overline{w(\ldots)}$$

$$- \frac{\partial}{\partial x}(\tfrac{1}{2}\bar{\rho}\bar{c}_0{}^2 u_0 + \tfrac{1}{2}\overline{\rho C^2 U} + \tfrac{1}{2}\overline{\rho C^2}u_0 + \overline{\rho U^2}u_0 + \overline{\rho UV}v_0 + \overline{\rho UW}w_0)$$

$$- \frac{\partial}{\partial y}(\ldots) - \frac{\partial}{\partial z}(\ldots).$$

Hence, on simplifying by (9) and (11),

$$\bar{\rho}\frac{D}{Dt}(\overline{\tfrac{1}{2}\rho C^2}/\bar{\rho}) = - \overline{U\frac{\partial p_1}{\partial x}} - \overline{V\frac{\partial p_1}{\partial y}} - \overline{W\frac{\partial p_1}{\partial z}}$$

$$- \overline{U\left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} + \frac{\partial P_{xz}}{\partial z}\right)} - \overline{V(\ldots)} - \overline{W(\ldots)}$$

$$- \frac{\partial}{\partial x}(\overline{\tfrac{1}{2}\rho C^2 U}) - \frac{\partial}{\partial y}(\overline{\tfrac{1}{2}\rho C^2 V}) - \frac{\partial}{\partial z}(\overline{\tfrac{1}{2}\rho C^2 W})$$

$$- \overline{\rho U^2}\frac{\partial u_0}{\partial x} - \overline{\rho V^2}\frac{\partial v_0}{\partial y} - \overline{\rho W^2}\frac{\partial w_0}{\partial z}$$

$$- \overline{\rho UV}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right) - \overline{\rho VW}\left(\frac{\partial w_0}{\partial y} + \frac{\partial v_0}{\partial z}\right) - \overline{\rho WU}\left(\frac{\partial u_0}{\partial z} + \frac{\partial w_0}{\partial x}\right). \quad (12)$$

Finally, on taking the mean value of (7) and using (9),

$$\bar{\rho}\frac{D}{Dt}(\overline{\rho E}/\bar{\rho}) = - \overline{p_1\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{dz}\right)} - \overline{P_{xx}\frac{\partial u}{\partial x}} - \overline{P_{yy}\frac{\partial v}{\partial y}} - \overline{P_{zz}\frac{\partial w}{\partial z}}$$

$$- \overline{P_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)} - \overline{P_{yz}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)} - \overline{P_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)}$$

$$+ \overline{\rho\epsilon_1} - \frac{\partial}{\partial x}((F_1)_x + \overline{\rho EU}) - \frac{\partial}{\partial y}((F_1)_y + \overline{\rho EV}) - \frac{\partial}{\partial z}((F_1)_z + \overline{\rho EW}). \quad (13)$$

Equations (9), (10), (12) and (13) determine the macroscopic behaviour of the gas.

4. Certain approximations valid in the present problem can be used to simplify these equations. First, the contribution to the viscous resistance arising from the mass motion can be neglected; hence the terms of (13) which involve the viscous stresses become equal to

$$\bar{\rho}\epsilon_2 = \mu\overline{\left\{\frac{2}{3}\left(\frac{\partial U}{\partial x}-\frac{\partial V}{\partial y}\right)^2+\frac{2}{3}\left(\frac{\partial V}{\partial y}-\frac{\partial W}{\partial z}\right)^2+\frac{2}{3}\left(\frac{\partial W}{\partial z}-\frac{\partial U}{\partial x}\right)^2\right.}$$
$$\overline{\left.+\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z}\right)^2+\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right)^2\right\}}, \qquad (14)$$

while those of (12) reduce to

$$-\bar{\rho}\epsilon_2-\overline{\frac{\partial}{\partial x}(UP_{xx}+VP_{xy}+WP_{xz})}-\overline{\frac{\partial}{\partial y}(\ldots)}-\overline{\frac{\partial}{\partial z}(\ldots)}.$$

The last three terms of this expression vanish when the degree of turbulence is uniform, and are in general small compared with the first term; we therefore neglect them. Again, in the problem which we shall consider the mass motion is purely radial; hence in (10) the effect of the viscous forces on the mass motion, which is small for a pure expansion, will be neglected.

In atmospheric problems it is found that there is approximate equipartition of turbulent kinetic energy for motion parallel to the three coordinate axes.* If we assume that this is true in the present problem, we have

$$\overline{\rho U^2}=\overline{\rho V^2}=\overline{\rho W^2}=p_2, \quad \text{say.}$$

Also, as the motion is radial, turbulence will be responsible for no "viscous" stresses; that is

$$\overline{\rho UV}=\overline{\rho VW}=\overline{\rho WU}=0.$$

With these simplifications, (10) and (12) reduce to

$$\bar{\rho}\frac{Du_0}{Dt}=\bar{\rho}X-\frac{\partial(\bar{p}_1+p_2)}{\partial x}, \qquad (15)$$

$$\bar{\rho}\frac{D}{Dt}(3p_2/2\bar{\rho})=\frac{p_2}{\bar{\rho}}\frac{D\bar{\rho}}{Dt}-\overline{\left(U\frac{\partial p_1}{\partial x}+V\frac{\partial p_1}{\partial y}+W\frac{\partial p_1}{\partial z}\right)}$$
$$-\bar{\rho}\epsilon_2-\frac{\partial(F_2)_x}{\partial x}-\frac{\partial(F_2)_y}{\partial y}-\frac{\partial(F_2)_z}{\partial z}, \qquad (16)$$

where

$$(F_2)_x=\overline{\tfrac{1}{2}\rho C^2U}, \qquad (F_2)_y=\overline{\tfrac{1}{2}\rho C^2V}, \qquad (F_2)_z=\overline{\tfrac{1}{2}\rho C^2W}, \qquad (17)$$

so that $(F_2)_x$, $(F_2)_y$, $(F_2)_z$ are the components of F_2, the vector of convection of turbulent energy.

In simplifying (13), we substitute

$$\rho E=p_1/(\gamma-1), \qquad (18)$$

which is the relation true for an ordinary gas. Then we obtain the relation

$$\bar{\rho}\frac{D}{Dt}\left(\frac{\bar{p}_1}{\bar{\rho}(\gamma-1)}\right)=\frac{\bar{p}_1}{\bar{\rho}}\frac{D\bar{\rho}}{Dt}+\overline{\rho\epsilon}_1+\bar{\rho}\epsilon_2-\bar{\rho}\epsilon_3$$
$$-\frac{\partial}{\partial x}((F_1)_x+(F_2)_x)-\frac{\partial}{\partial y}((F_1)_y+(F_2)_y)-\frac{\partial}{\partial z}((F_1)_z+(F_2)_z), \qquad (19)$$

* A summary of the experimental evidence is given on pp. 210–213 of Brunt's book quoted above.

where

$$\bar{\rho}\epsilon_3 = \overline{p_1\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)}, \tag{20}$$

$$(F_3)_x = \overline{\rho EU}, \qquad (F_3)_y = \overline{\rho EV}, \qquad (F_3)_z = \overline{\rho EW}, \tag{21}$$

so that $\bar{\rho}\epsilon_3$ is the rate of conversion of thermal energy into turbulent kinetic energy, and $(F_3)_x$, $(F_3)_y$, $(F_3)_z$ are the components of F_3, the vector measuring the rate of convection of thermal energy.

Equations (9), (15), (16) and (19) can be further simplified by using the facts that c_0, F_1, etc., are directed radially, and all the variables depend only on t and r, the distance from the centre of the star. We take c_0, F_1, etc., as positive if directed outward, negative if inward. In evaluating the mean values

$$\overline{U\frac{\partial p_1}{\partial x}+V\frac{\partial p_1}{\partial y}+W\frac{\partial p_1}{\partial z}}, \qquad \overline{p_1\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)}$$

appearing in (16) and (20), we ignore the variations of p_1 due to the turbulence, as in § 2. Then if C_r is the mean value of the radial component of the turbulent velocity, we have

$$\overline{U\frac{\partial p_1}{\partial x}+V\frac{\partial p_1}{\partial y}+W\frac{\partial p_1}{\partial z}} = C_r\frac{\partial p_1}{\partial r}, \tag{22}$$

$$\overline{p_1\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)} = \frac{p_1}{r^2}\frac{\partial}{\partial r}(r^2C_r). \tag{23}$$

It will in general be possible in subsequent work to omit the mean value symbol in $\bar{\rho}$, $\bar{p}_1$, etc., the symbols being understood to refer to mean values.

It is convenient at this point to express the different variables as functions, not of r and t, but of m and t, where m is the mass enclosed by a sphere of radius r concentric with the star. The effect of this change is that the differential coefficient D/Dt following the mean motion becomes the ordinary differential coefficient $\partial/\partial t$, and $\partial/\partial r$ is replaced by $4\pi\rho r^2\partial/\partial m$. Hence, for example, the new form of (9) is

$$\begin{aligned}\frac{1}{\rho^2}\frac{\partial\rho}{\partial t} &= -4\pi\frac{\partial}{\partial m}(r^2c_0)\\ &= -\frac{\partial}{\partial m}\left(4\pi r^2\frac{\partial r}{\partial t}\right),\end{aligned}$$

which can be integrated with respect to t to give the known relation

$$\frac{1}{\rho} = \frac{\partial}{\partial m}(\tfrac{4}{3}\pi r^3). \tag{24}$$

Again, since the only force on the matter is gravity, producing an acceleration Gm/r^2, where G is the gravitational constant, (15) becomes

$$\frac{\partial^2 r}{\partial t^2} \equiv \frac{\partial c_0}{\partial t} = -\frac{Gm}{r^2} - 4\pi r^2\frac{\partial(p_1+p_2)}{\partial m}; \tag{25}$$

while (16) and (19) become

$$\frac{\partial}{\partial t}\left(\frac{3p_2}{2\rho}\right)=\frac{p_2}{\rho^2}\frac{\partial\rho}{\partial t}-4\pi r^2C_r\frac{\partial p_1}{\partial m}-\epsilon_2-\frac{\partial}{\partial m}(4\pi r^2F_2), \tag{26}$$

$$\frac{\partial}{\partial t}\left(\frac{p_1}{(\gamma-1)\rho}\right)=\frac{p_1}{\rho^2}\frac{\partial\rho}{\partial t}+\epsilon_1+\epsilon_2-\epsilon_3-\frac{\partial}{\partial m}(4\pi r^2(F_1+F_3)). \tag{27}$$

Also (20) now becomes

$$\epsilon_3=p_1\frac{\partial}{\partial m}(4\pi r^2C_r). \tag{28}$$

By (2), the value of C_r is

$$C_r=-4\pi\rho r^2lq\left(\frac{1}{T}\frac{\partial T}{\partial m}-\frac{(\gamma-1)}{\gamma p_1}\frac{\partial p_1}{\partial m}\right). \tag{29}$$

5. Consider first the "equilibrium" configuration of the star, in which $c_0=0$. For this, (25), (26) and (27) are replaced by

$$\frac{\partial(p_1+p_2)}{\partial m}=-\frac{Gm}{4\pi r^4}, \tag{30}$$

$$4\pi r^2C_r\frac{\partial p_1}{\partial m}+\epsilon_2+\frac{\partial}{\partial m}(4\pi r^2F_2)=0, \tag{31}$$

$$\frac{\partial}{\partial m}(4\pi r^2(F_1+F_3))=\epsilon_1+\epsilon_2-\epsilon_3. \tag{32}$$

Of these, (30) and (32) are generalisations of equations valid in the absence of convection, while (31) is a completely new equation, determining the degree of turbulence. The first of the terms on the left of (31) can be evaluated using equation (29). The term $\rho\epsilon_2$ is given by (14); as the cofactor of μ on the right of this is of the same order as the square of the velocity-gradient, one might expect $\rho\epsilon_2$ to be of the same order as $\mu q^2/l^2$, where q, l have the same meanings as in § 2. It will, however, be considerably larger when, as we have supposed, the convection is turbulent, since in this case the majority of the dissipation of energy occurs in the small eddies in which the velocity-gradient is much larger than q/l. We can estimate its magnitude by observing that, as a mass of gas is supposed to be brought to rest after travelling a distance l, in the absence of factors maintaining the convection a large part of the turbulent energy, which is of order ρq^2 per unit volume, would be destroyed in the time, of order l/q, which the mass takes to travel this distance. Hence as $\rho\epsilon_2$ is equal to the rate of conversion of turbulent kinetic energy into thermal energy, we can set

$$\rho\epsilon_2=A\rho q^3/l, \tag{33}$$

where A is a constant of order unity.

The third term on the left of (31) represents the effect of the flow of turbulence from point to point, and vanishes on integration over the whole mass of the star. We therefore neglect it in this work.

On use of (29) and (33), (31) becomes

$$16\pi^2\rho r^4 lq\frac{\partial p_1}{\partial m}\left(\frac{1}{T}\frac{\partial T}{\partial m}-\frac{(\gamma-1)}{\gamma p_1}\frac{\partial p_1}{\partial m}\right)=\frac{Aq^3}{l}. \qquad (34)$$

Only this one equation connecting l and q can be derived by the present method. The actual value of l is unknown ; it is in general of the order of the linear dimensions of the unstable region, but smaller.

To give us some idea of the magnitude of q, we consider, following Biermann,* a star of the magnitude of the Sun, possessing a central convective zone. It appears from work by Biermann, Rosseland and Steensholt† that the convective zone is extensive; under suitable conditions there is convection at all points whose distance from the centre is less than, say, one-third of the radius. We therefore consider a point at a distance 10^{10} cm. from the centre as typical of the convective zone.

The total outward flux F of energy per unit area by radiation, convection, etc., is made up of F_1, F_2, F_3 and the rate p_1C_r at which work is done by gas across a sphere concentric with the star. Of these parts, F_1, F_3 and p_1C_r are all positive ; F_2 can be positive or negative but is in any case smaller than the others. Hence $F_3 < F$, or, using (3),

$$-4\pi\rho^2r^2lqc_v\left(\frac{\partial T}{\partial m}-\frac{T(\gamma-1)}{\gamma p_1}\frac{\partial p_1}{\partial m}\right)<F.$$

Thus, by (34)

$$\frac{Aq^3}{l} < -4\pi r^2F\frac{\partial p_1}{\partial m}\Big/\rho c_v T$$

$$< -F(\gamma-1)\frac{\partial p_1}{\partial r}\Big/p_1\rho.$$

Now, if $r=10^{10}$ cm., F cannot be greater than 3×10^{12} ergs/cm.^{2} sec., its value if all the sources of energy lie closer to the centre ; also $\gamma<\frac{5}{3}$. We set $-\frac{1}{p_1}\frac{\partial p_1}{\partial r}$ and ρ equal respectively to 10^{-10} cm.$^{-1}$ and 36 grs./cm.^{3} (their values for a polytrope of index 3 are $1{\cdot}2\times10^{-10}$ cm.$^{-1}$ and 48 grs./cm.^{3} respectively). Then

$$\epsilon_2=\frac{Aq^3}{l}<5{\cdot}5 \text{ cm.}^2/\text{sec.}^3.$$

Since F_3 will be an appreciable part of F, the actual value of ϵ_2 will not be small compared with this upper limit. As the mean value of ϵ_1 for the whole mass of the Sun is about 2 cm.^{2}/sec.^{3}, we shall assume that $\epsilon_1=\epsilon_2$ approximately.

As the dimensions of the unstable region are of order 2×10^{10} cm., l may be estimated as 10^9 cm. Hence, on setting $A=1$,

$$q<1{\cdot}8\times10^3 \text{ cm./sec.}$$

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* *Zeit. für Astrophys.*, 5, 117, 1932.

† *Zeit. für Astrophys.*, 4, 255, 1932 ; 5, 140, 1932. See also the Appendix to the present paper.

Using this in (34), and setting $p_1 = 8{\cdot}26 \times 10^7 \rho T$, as in a gas of molecular weight unity, we find

$$-\left(\frac{\partial T}{\partial r} - \frac{(\gamma-1)}{\gamma p}\frac{\partial p}{\partial r}\right) < 4 \times 10^{-10} \text{ deg./cm.}$$

Since the value of $-\partial T/\partial r$ may be estimated as 10^{-3} deg./cm., the divergence from adiabatic equilibrium is very slight.

Similar results hold for all stars.

6. Consider now small oscillations about the position of equilibrium. Let the displacement of a given layer from the centre be r in the equilibrium position, and $r+\delta r$ in the disturbed, with a similar notation for the other variables; and let the products of small quantities be neglected. Then the equations governing the equilibrium state are (24), (30), (31) and (32). The corresponding equations governing the disturbed state are derived on applying a variation to (24), (25), (26) and (27); these can be put in the forms

$$\frac{\partial}{\partial m}(4\pi r^2 \delta r) = -\frac{\delta\rho}{\rho^2} \tag{35}$$

$$\frac{\partial^2 \delta r}{\partial t^2} = -4\pi r^2 \frac{\partial(\delta p_1 + \delta p_2)}{\partial m} + \frac{4Gm}{r^3}\delta r$$

$$\frac{3}{2\rho}\frac{\partial \delta p_2}{\partial t} - \frac{5p_2}{2\rho^2}\frac{\partial \delta\rho}{\partial t} = -\delta\left(4\pi r^2 C_r \frac{\partial p_1}{\partial m} + \epsilon_2 + \frac{\partial}{\partial m}(4\pi r^2 F_2)\right)$$

$$\frac{1}{\rho}\frac{\partial \delta p_1}{\partial t} - \frac{\gamma p_1}{\rho^2}\frac{\partial \delta\rho}{\partial t} = (\gamma-1)\delta\left(\epsilon_1 + \epsilon_2 - \epsilon_3 + \frac{\partial}{\partial m}(4\pi r^2(F_1+F_3))\right)$$

We suppose that for any oscillation δr, $\delta\rho$, etc., are each proportional to $e^{\iota\sigma t}$, where in interpreting our results we take the real part of any variable. Then the last three equations become

$$\left(\sigma^2 + \frac{4Gm}{r^3}\right)\delta r = 4\pi r^2 \frac{\partial}{\partial m}(\delta p_1 + \delta p_2), \tag{36}$$

$$\iota\sigma\left(\delta p_2 - \frac{5p_2}{3\rho}\delta\rho\right) = -\tfrac{2}{3}\rho\delta\left(4\pi r^2 C_r \frac{\partial p_1}{\partial m} + \epsilon_2 + \frac{\partial}{\partial m}(4\pi r^2 F_2)\right), \tag{37}$$

$$\iota\sigma\left(\delta p_1 - \frac{\gamma p_1}{\rho}\delta\rho\right) = \rho(\gamma-1)\delta\left(\epsilon_1 + \epsilon_2 - \epsilon_3 + \frac{\partial}{\partial m}(4\pi r^2(F_1+F_3))\right). \tag{38}$$

We require solutions of equations (35), (36), (37) and (38) such that δp_1, δp_2, $\delta\rho$ vanish at the boundary, where $m = M$, the mass of the star, and δr vanishes at the centre, where $m = 0$. The values of σ giving such solutions are in general complex. In using the equations we can suppose the factor $e^{\iota\sigma t}$ divided out from δr, $\delta\rho$, etc., which therefore depend only on m.

Adiabatic oscillations of the star are defined as such that the right-hand sides of (37) and (38), which represent the effect of the generation and transport of energy, are set equal to zero. For such oscillations σ is real, and we can take δr, $\delta\rho$, etc., as real. The actual values of σ when the right-

hand sides of (37) and (38) are not set equal to zero are close to the values for adiabatic oscillations, since the terms on the right-hand sides are small compared with those on the left throughout the major part of the star.*

Let σ refer to an actual oscillation, σ_a to the corresponding adiabatic oscillation ; then

$$\sigma = \sigma_a + \sigma', \tag{39}$$

where σ' is small. Let the values of δr, $\delta\rho$, etc., for the adiabatic oscillation be distinguished by a suffix a ; then δr_a, $\delta\rho_a$, etc., satisfy equations of the forms (35) and (36), together with

$$\delta p_{1a} = \gamma p_1 \delta\rho_a/\rho, \qquad \delta p_{2a} = 5p_2\delta\rho/3\rho. \tag{40}$$

Also we can suppose that δr, $\delta\rho$, etc., are very nearly equal to δr_a, $\delta\rho_a$, etc.

We have, from (36),

$$\iota\sigma(\sigma^2 - \sigma_a{}^2)\int_0^M \delta r\delta r_a dm = \iota\sigma\int_0^M \left[\left(\sigma^2 + \frac{4Gm}{r^3}\right)\delta r\delta r_a - \left(\sigma_a{}^2 + \frac{4Gm}{r^3}\right)\delta r_a\delta r\right]dm$$

$$= \iota\sigma\int_0^M \left[4\pi r^2\delta r_a\left(\frac{\partial\delta p_1}{\partial m} + \frac{\partial\delta p_2}{\partial m}\right) - 4\pi r^2\delta r\left(\frac{\partial\delta p_{1a}}{\partial m} + \frac{\partial\delta p_{2a}}{\partial m}\right)\right]dm.$$

On the left-hand side of this equation use (39), and neglect products of small quantities ; and on the right-hand side integrate by parts, and use (35), (37), (38) and (40). Then

$$2\iota\sigma_a{}^2\sigma'\int_0^M (\delta r_a)^2 dm = \iota\sigma\int_0^M \left[\frac{\delta\rho_a}{\rho^2}(\delta p_1 + \delta p_2) - \frac{\delta\rho}{\rho^2}(\delta p_{1a} + \delta p_{2a})\right]dm$$

$$= \int_0^M \frac{\delta\rho_a}{\rho}\left\{(\gamma - 1)\delta\left(\epsilon_1 + \epsilon_2 - \epsilon_3 - \frac{\partial}{\partial m}(4\pi r^2(F_1 + F_3))\right)\right.$$

$$\left. - \tfrac{2}{3}\delta\left(4\pi r^2 C_r\frac{\partial p_1}{\partial m} + \epsilon_2 + \frac{\partial}{\partial m}(4\pi r^2 F_2)\right)\right\}dm.$$

For stability the real part of $\iota\sigma$ must be negative ; hence the right-hand side of this equation must have its real part negative. In evaluating the right-hand side we can assume, to the present order of approximation, that δr, $\delta\rho$, δp_1 and δp_2 have their adiabatic values. Hence the condition of stability is

$$\int_0^M \frac{\delta\rho}{\rho}\left\{(\gamma - 1)\delta\left(\epsilon_1 + \epsilon_2 - \epsilon_3 - \frac{\partial}{\partial m}(4\pi r^2(F_1 + F_3))\right)\right.$$

$$\left. - \tfrac{2}{3}\delta\left(4\pi r^2 C_r\frac{\partial p_1}{\partial m} + \epsilon_2 + \frac{\partial}{\partial m}(4\pi r^2 F_2)\right)\right\}dm < 0, \tag{41}$$

where δr, $\delta\rho$, etc., are supposed to satisfy equations (35), (36) and (40).

7. The condition of stability can be simplified, using the following considerations. Turbulence only affects an adiabatic oscillation through the term δp_2 in (36), and this is negligible compared with δp_1. Hence

* Since $\delta r/r$, $\delta\rho/\rho$, $\delta p_1/p_1$, $\delta p_2/p_2$, etc., are all of the same order, the ratio of terms on the right of (37) to terms on the left is of order $\rho\epsilon_2/\sigma p_2$; this, for the Sun, is of order 2×10^{-3}. A similar result holds for (38) (*cf.* Eddington, *Internal Constitution of the Stars*, p. 197).

results obtained for adiabatic oscillations when convection is absent can be applied to the present problem. In the principal oscillation—which is the one of importance in stability work—it is known that $\delta r/r$, $\delta\rho/\rho$ vary but little with m in the central regions, though increasing rapidly in the outer layers. As we are considering convection confined to the central regions, in evaluating the contribution to the integral (41) from terms depending on convection we can regard $\delta\rho/\rho$ as independent of m. Thus, if the outer limit of convection corresponds to $m = m_0$, we shall have

$$\begin{aligned}\int_0^{m_0} \frac{\delta\rho}{\rho}\delta\frac{\partial}{\partial m}[4\pi r^2\{(\gamma-1)(F_1+F_3)+\tfrac{2}{3}F_2\}]dm \\ =\frac{\delta\rho}{\rho}\delta\left[4\pi r^2\{(\gamma-1)(F_1+F_3)+\tfrac{2}{3}F_2\right]_0^{m_0} \\ =\frac{\delta\rho}{\rho}\delta[4\pi r^2(\gamma-1)F_1]_{m=m_0}.\end{aligned}$$

Hence the terms involving F_2, F_3 can be omitted from (41). The value of the integral does not, in fact, depend on the magnitude of the transport of energy at points inside the convective region, but only on its value at the boundary of the region. Again

$$\begin{aligned}\int_0^M \frac{\delta\rho}{\rho}\delta\epsilon_3 dm &= \frac{\delta\rho}{\rho}\delta\int_0^{m_0}\epsilon_3 dm \\ &= \frac{\delta\rho}{\rho}\delta\int_0^{m_0} p_1\frac{\partial}{\partial m}(4\pi r^2 C_r)dm \\ &= -\frac{\delta\rho}{\rho}\delta\int_0^{m_0} 4\pi r^2 C_r\frac{\partial p_1}{\partial m}dm \\ &= -\int_0^M\frac{\delta\rho}{\rho}\delta\left(4\pi r^2 C_r\frac{\partial p_1}{\partial m}\right)dm.\end{aligned}$$

Thus (41) becomes

$$\int_0^M \frac{\delta\rho}{\rho}\left\{(\gamma-1)\delta\left(\epsilon_1-\frac{\partial}{\partial m}(4\pi r^2F_1)\right)+(\gamma-\tfrac{5}{3})\delta\left(\epsilon_2+4\pi r^2C_r\frac{\partial p_1}{\partial m}\right)\right\}dm<0. \quad (42)$$

This differs from the condition of stability when turbulence is absent only by reason of the terms involving

$$\delta\left(\epsilon_2+4\pi r^2C_r\frac{\partial p_1}{\partial m}\right).$$

By (29) and (33), this expression is equal to

$$\delta\left(\frac{Aq^3}{l}-16\pi\rho r^4 lq\left(\frac{1}{T}\frac{\partial T}{\partial m}-\frac{(\gamma-1)}{\gamma p_1}\frac{\partial p_1}{\partial m}\right)\frac{\partial p_1}{\partial m}\right).$$

Now in the central regions of the star $\delta\rho/\rho = -3\delta r/r$ approximately for the principal oscillation. Also if, as we may expect, l varies in proportion with the linear dimensions of a mass of gas,

$$\delta l/l = \delta r/r = -\delta\rho/3\rho.$$

Again, by (40)

$$\delta\left(\frac{1}{T}\frac{\partial T}{\partial m}-\frac{(\gamma-1)}{\gamma p_1}\frac{\partial p_1}{\partial m}\right)=\delta\frac{\partial}{\partial m}\log\left(T/p_1^{\frac{\gamma-1}{\gamma}}\right)$$

$$=\frac{\partial}{\partial m}\delta\log\left(p_1^{1/\gamma}/\rho\right)$$

$$=\frac{\partial}{\partial m}\left(\frac{\delta p_1}{\gamma p_1}-\frac{\delta\rho}{\rho}\right)=0,$$

and by (30) and (36), approximately

$$\delta\left(\frac{\partial p_1}{\partial m}\right)\Big/\frac{\partial p_1}{\partial m}=-\left(4+\frac{\sigma^2r^3}{Gm}\right)\frac{\delta r}{r}$$

$$=\left(4+\frac{\sigma^2r^3}{Gm}\right)\frac{\delta\rho}{3\rho}.$$

Finally, as $\delta p_2/p_2=5\delta\rho/3\rho$, and p_2 is proportional to ρq^2,

$$\delta q/q=\delta\rho/3\rho.$$

Combining all these results, we have

$$\delta\left(\epsilon_2+4\pi r^2C_r\frac{\partial p_1}{\partial m}\right)$$

$$=\frac{Aq^3}{l}\cdot\frac{4\delta\rho}{3\rho}-16\pi\rho r^4lq\left(\frac{1}{T}\frac{\partial T}{\partial m}-\frac{(\gamma-1)}{\gamma p_1}\frac{\partial p_1}{\partial m}\right)\frac{\partial p_1}{\partial m}\left(3+\frac{\sigma^2r^3}{Gm}\right)\frac{\delta\rho}{3\rho}$$

$$=\frac{Aq^3}{3l}\cdot\frac{\delta\rho}{\rho}\left(1-\frac{\sigma^2r^3}{Gm}\right)$$

by (34). Hence the contribution to the integral (42) from turbulence is

$$-\left(\tfrac{5}{3}-\gamma\right)\int_0^M\epsilon_2\left(\frac{\delta\rho}{\rho}\right)^2\left(1-\frac{\sigma^2r^3}{Gm}\right)dm;$$

or, if we set $\epsilon_2=\epsilon_1$, as was suggested in § 5, approximately

$$-\left(\tfrac{5}{3}-\gamma\right)\left(1-\left(\frac{\sigma^2r^3}{Gm}\right)_c\right)\int_0^M\epsilon_1\left(\frac{\delta\rho}{\rho}\right)^2dm, \tag{43}$$

the suffix c implying that the quantity in question is evaluated at the centre of the star. Since, by relation (21) of I, $(\sigma^2r^3/Gm)_c$ is always less than unity, turbulence increases the stability of a star.

The magnitude of the remaining part of the integral (42) may be estimated by the methods used in I, even though in that paper it was assumed throughout that there is no convection. This assumption affected the work in two ways. It was supposed that the radiative transport of energy F_1 is the whole of the transport, and that in the equilibrium state

$$4\pi r^2F_1=\int_0^m\epsilon_1dm;$$

this is also true in the outer layers of a star with a central convection zone, and we have seen above that the value of the flow of energy in the central regions has no appreciable effect on the value of the integral (42). Again, it was supposed that the actual temperature gradient never exceeds the adiabatic; this is true for the present model in the outer layers of the star, and in the central region the temperature gradient is almost exactly equal to the adiabatic; hence again no appreciable error is involved.

8. Let us now consider special values of γ. As in I, we shall assume that the opacity satisfies Kramers' law

$$\kappa \propto \rho/T^{7/2}.$$

We also suppose that the law of generation of energy is of the form

$$\epsilon_1 \propto \rho T^{\beta}, \tag{44}$$

so that

$$\frac{\delta\epsilon_1}{\epsilon_1} = \frac{\delta\rho}{\rho} + \beta\frac{\delta T}{T}$$
$$= \frac{\delta\rho}{\rho}(1 + \beta(\gamma - 1)).$$

If $\gamma = \frac{5}{3}$, (43) vanishes, and convection does not affect stability. Also it is shown in the Appendix that in this case

$$\int_0^M \frac{\delta\rho}{\rho}\delta\frac{d}{dm}(4\pi r^2 F_1)dm > 300\int_0^M \epsilon_1\left(\frac{\delta\rho}{\rho}\right)^2 dm.$$

Hence the condition of stability (42) is satisfied if

$$(\gamma - 1)(1 + \beta(\gamma - 1) - 300)\int_0^M \epsilon_1\left(\frac{\delta\rho}{\rho}\right)^2 dm < 0,$$

i.e. if $\beta < 448$. It is difficult to imagine a mode of generation of energy for which this is not satisfied.

If $\gamma = 10/7$, the value of $(\sigma^2 r^3/Gm)_c$ is taken equal to that obtained by Eddington for the standard model, as in I. Then (43) becomes equal to

$$0{\cdot}22\int_0^M \epsilon_1\left(\frac{\delta\rho}{\rho}\right)^2 dm,$$

and it was shown in I that

$$\int_0^M \frac{\delta\rho}{\rho}\delta\frac{d}{dm}(4\pi r^2 F_1)dm > 8{\cdot}25\int_0^M \epsilon_1\left(\frac{\delta\rho}{\rho}\right)^2 dm.$$

Hence there is stability if

$$(\gamma - 1)(1 + \beta(\gamma - 1) - 8{\cdot}25) - 0{\cdot}22 < 0,$$

i.e. if $\beta < 18{\cdot}1$.

Similarly for the case $\gamma = 40/29$, using the results of I, we find

$$(\gamma - 1)(1 + \beta(\gamma - 1) - 2{\cdot}15) - 0{\cdot}28 < 0,$$

or $\beta < 4{\cdot}98$. It is to be noted that convection has very little effect on the stability condition except when γ is nearly equal to 4/3.

Atkinson's * theory of generation of energy by nuclear penetration corresponds to a law of generation of the form (44), with β about 20. It has usually been supposed that stars built on Atkinson's model are unstable ; we see from the above that they are actually stable if γ is greater than, say, 1·44. If radiation, which in adiabatic changes behaves like a gas with $\gamma = 4/3$, supplies a large part of the total pressure, the star is unstable. Thus massive stars should be unstable, smaller stars stable. This agrees with the hypothesis that Cepheid variables are stars whose small oscillations increase in amplitude until second-order effects arise which prevent further increase.†

If this hypothesis is valid, the second-order effects may be expected to be responsible for the fact that maximum brightness occurs at the time of maximum outward velocity, not at that of minimum extension. This implies that they include an unbalanced warming of the layers under the photosphere while the star is in a compressed state. This unbalanced warming may possibly result from a convection of heat up to a layer closer to the surface than is normally reached. If so, the difference between a Cepheid and a long-period variable may be that in the latter convection is more extensive, and the masses in motion are appreciable portions of the whole star, whose motion may extend up to the surface.

These are, however, only speculations. The important fact is that generation of energy by atomic processes does not necessarily produce vibrational instability. A wide field is open for further work on the properties of models in which the sources of energy are strongly concentrated towards the centre.

9. In conclusion one word is necessary. Use of the "free-path" method in work on turbulence is liable to result in some error. For example, if the variation of turbulent energy during an oscillation is adiabatic, the time of a free path is large compared with the period of the oscillation. It is therefore doubtful whether exact equipartition of turbulent energy for motion parallel to the three co-ordinate axes can be assumed. However, it is not likely that the errors are sufficient to upset our conclusion that convection has only a small effect on the stability, and so our fundamental results are unaffected.

Appendix

The degree of central condensation of a star increases with the mean throughout the star of the equivalent polytropic index n, defined by

$$d \log p_1 / d \log \rho = 1 + 1/n.$$

In the central regions of a star with a convective core, the polytropic index is approximately $1/(\gamma - 1)$; in the outer regions there is radiative equilibrium. Since a concentration of sources towards the centre decreases the polytropic

* *Astrophys. Jour.*, **73**, 250, 308, 1931.

† It is, perhaps, significant that Eddington found low values of γ, in the neighbourhood of 1·4, for Cepheids, on the pulsation theory.

index in the outer regions, the maximum degree of uniformity of density occurs when the sources are all within the convection zone, and the polytropic index within that zone has its least value 3/2, corresponding to $\gamma = 5/3$. This last implies that the gas is monatomic, and the ratio of radiation pressure to gas pressure is negligible.

Consider the star most uniform in density. With the notation used above,

$$\frac{dp_1}{dr} = -\frac{Gm\rho}{r^2},$$

$$\frac{dm}{dr} = 4\pi\rho r^2.$$

Also, if the coefficient of opacity is taken to be $a\rho/T^{7/2}$, and the luminosity of the star is L, in the outer regions

$$\frac{d}{dr}(\tfrac{1}{3}aT^4) = -\frac{aL\rho^2}{4\pi cr^2T^{7/2}},$$

where a is the radiation constant and c is now the velocity of light; while in the convection zone

$$\rho \propto T^{3/2}.$$

If r, T, ρ, m are taken proportional to suitably chosen non-dimensional variables, x, y, z, w, these equations can be put in the forms

$$\frac{d(yz)}{dx} = -\frac{5wz}{2x^2}, \tag{45}$$

$$\frac{dw}{dx} = zx^2, \tag{46}$$

and in the centre of the star,

$$z = y^{3/2}; \tag{47}$$

while in the outer regions

$$\frac{dy}{dx} = -\frac{kz^2}{x^2y^{13/2}}, \tag{48}$$

where k is a numerical constant. We require a set of solutions of these equations such that $w = 0$, and y and z are finite, when $x = 0$; that y, z, w and their derivatives with respect to x are continuous at the junction of the regions in which (47) and (48) hold; and that y and z vanish at the boundary.

The solution is by trial. In the central region the equation connecting y and x is Emden's equation of order 3/2. Hence, if we assume that $y = 1$ when $x = 0$, the values of y, z and w in the region can be obtained from tables of Emden functions. We now suppose that the outer boundary of convection corresponds to some value $x = x_0$, and adjust the value of k so as to make the value of dy/dx given by (48) continuous with that derived from Emden's equation. Then by trial we find what must be the value of x_0 so that y and z may vanish together. After trial solutions have been

obtained for two or three values of k, we begin new solutions by interpolating between these. In actual integration it is found convenient to use the variable Z, where $z = Z^3$, in place of z, since dZ/dx varies much more slowly than dz/dx. It is found that $x_0 = 1{\cdot}188$, and that the boundary corresponds to $x = 7{\cdot}027$ approximately. A table of values of y, z and w is given below.

Since the boundary of a polytrope of index 3/2 is given by $x = 3{\cdot}654$, there is a considerable divergence from such a polytrope. The ratio of central to mean density is equal to the boundary value of $x^3/3w$, which is about 37. This lies between the values 24 and 54 for the same ratio for polytropes of index 2·5 and 3 respectively. Thus the minimum value of this ratio for gaseous stars is not much less than the value for a polytrope of index 3 ; the ratio may rise to decidedly greater values if γ is smaller.

For stability work we require to know the variation of $\delta r/r$, $\delta\rho/\rho$ with r for the principal adiabatic oscillation. By equations (35), (36), (30) and (40), neglecting the small terms p_2, δp_2, we have

$$\frac{d(r^2\delta r)}{r^2 dr} = -\frac{\delta\rho}{\rho},$$

$$\frac{d\delta p_1}{dp_1} = -\frac{\delta r}{r}\left(4 + \frac{\sigma^2 r^3}{Gm}\right),$$

$$\frac{\delta p_1}{p_1} = \frac{5\delta\rho}{3\rho};$$

and so

$$\frac{d}{x^2 dx}\left(x^3 \frac{\delta r}{r}\right) = -\frac{\delta\rho}{\rho},$$

$$\frac{5}{3}\frac{d}{d(yz)}\left(yz\frac{\delta\rho}{\rho}\right) = -\frac{\delta r}{r}\left(4 + \frac{\Sigma^2 x^3}{w}\right),$$

where Σ^2 is defined by the equation

$$\Sigma^2 x^3/w = \sigma^2 r^3/Gm.$$

Hence finally, using (45),

$$\frac{d}{dx}\left(\frac{\delta r}{r}\right) = -\frac{1}{x}\left(\frac{\delta\rho}{\rho} + \frac{3\delta r}{r}\right), \tag{49}$$

$$\frac{d}{dx}\left(\frac{\delta\rho}{\rho}\right) = \frac{x}{y}\left\{\frac{3}{2}\left(\Sigma^2 + \frac{4w}{x^3}\right)\frac{\delta r}{r} + \frac{5}{2}\frac{w}{x^3}\frac{\delta\rho}{\rho}\right\}. \tag{50}$$

We require a solution of these equations such that $\delta r/r$, $\delta\rho/\rho$ are finite at the centre and at the boundary of the model. The value of Σ^2 for which such a solution exists is found by trial to lie between 0·086 and 0·0868. The values of $\delta r/r$ and $-\delta\rho/\rho$ corresponding to the latter value are set out in the last two columns of the table ; the initial values are assumed to be $\delta r/r = 1$, $\delta\rho/\rho = -3$ when $x = 0$.

x	y	z	w	$\delta r/r$	$-\delta\rho/\rho$
0	1	1	0	1	3
0·4	·9737	·9607	0·0208	1·0060	3·0299
0·8	·8984	·8518	0·1625	1·0245	3·1251
1·2	·7840	·6934	0·4653	1·0583	3·3029
1·6	·6611	·5075	0·9340	1·1120	3·6025
2·0	·5508	·3305	1·4631	1·1936	4·0731
2·4	·4546	·1971	1·9571	1·3125	4·7851
2·8	·37205	·1105	2·3585	1·4807	5·8336
3·2	·30268	$5·905 \times 10^{-2}$	2·6516	1·7160	7·3638
3·6	·24479	$3·047 \times 10^{-2}$	2·8491	2·0442	9·5678
4·0	·19623	$1·509 \times 10^{-2}$	2·9743	2·4945	12·707
4·4	·15554	$7·161 \times 10^{-3}$	3·0498	3·1036	17·111
4·8	·12118	$3·199 \times 10^{-3}$	3·0910	3·9298	23·223
5·2	·09186	$1·302 \times 10^{-3}$	3·1119	5·045	31·639
5·6	·06665	$4·597 \times 10^{-4}$	3·1212	6·511	42·485
6·0	·04477	$1·262 \times 10^{-4}$	3·1236	8·320	53·589
6·2	...	...	...	9·271	54·849
6·4	·02563	$2·058 \times 10^{-5}$	3·1237	...	...
6·8	·00873	$6·22 \times 10^{-7}$	3·1237	...	...
7·0	·00100	$5·4 \times 10^{-10}$	3·1237	...	...

It appears from this that the value of $\delta r/r$ at the boundary is at least 14. By (50) this corresponds to a boundary value of $\delta\rho/\rho$ equal to

$$-\frac{3\delta r}{5r}\left(4+\frac{\Sigma^2 x^3}{w}\right)=-114,$$

i.e. 38 times the central value. Thus the mean values Δ, Δ' used in I may be estimated as given by

$$\Delta=(20\delta\rho/\rho)_c^2, \qquad \Delta'=(1·05\delta\rho/\rho)_c^2;$$

whence, by the methods used in that paper,

$$\int_0^M \frac{\delta\rho}{\rho}\delta\frac{d}{dm}(4\pi r^2 F_1)dm > 2600\int_0^M \epsilon_1\left(\frac{\delta\rho}{\rho}\right)^2 dm.$$

If the sources of energy are not all inside the convective region, the model will be more strongly condensed toward the centre, and $\delta\rho/\rho$ will increase more between centre and boundary; on the other hand, Δ' will be greater than $(1·05\delta\rho/\rho)_c^2$. In any case, Δ' is not likely to be greater than $(3\delta\rho/\rho)_c^2$; hence we shall certainly have

$$\int_0^M \frac{\delta\rho}{\rho}\delta\frac{d}{dm}(4\pi r^2 F_1)dm > 300\int_0^M \epsilon_1\left(\frac{\delta\rho}{\rho}\right)^2 dm.$$

Swansea:
1935 *July* 18.

ON THE GALACTIC ROTATION OF THE GLOBULAR CLUSTERS.

Henri Mineur, D.-ès-Sc.

1. The motion of the globular clusters cannot be studied, at present, otherwise than through their radial velocities. Hitherto the globular clusters system as a whole has been supposed not to revolve appreciably around the centre of the galaxy, and one has simply determined the translation of the Sun with reference to this system.

I intend to show in this note that the globular clusters seem, on the contrary, to revolve with great velocity around the centre of the galaxy.

2. We shall use the following units :—

length . . . . 10,000 parsecs,
velocity . . . km./sec.

Let us suppose the centre, O, of the galaxy to be 10,000 parsecs distant towards longitude 325°, and let us take galactic axes which have this centre O as origin ; xOy is the galactic plane and Ox is in the equatorial plane. With these axes the co-ordinates (x_0, y_0) of the Sun, S, are

$$x_0 = -0{\cdot}82, \qquad y_0 = +0{\cdot}57.$$

Let X, Y, Z stand for the velocity of the Sun with reference to O. Let M denote a globular cluster, r its distance from Oz and z its distance from the galactic plane. We shall try to represent the radial velocities of the globular clusters supposing that each cluster (its residual velocity excepted) revolves around the axis Oz of the galaxy with a velocity V in the form

$$V = ar + br^2 + crz^2, \tag{1}$$

where a, b and c denote constants.

The formula (1) must be considered a mere formula of interpolation whose value, purely empirical and approximate, is limited to an area which we will presently determine. One must indeed expect the angular velocity of rotation to decrease when the distance from O and from the xOy plane increases (terms with the coefficients b and c), this last diminution is symmetrical with regard to the xOy plane.

If α, β, γ are the direction cosines of the direction SM, each globular cluster gives an equation of the form

$$\rho = -\alpha X - \beta Y - \gamma Z + a(\beta x - \alpha y) + br(\beta x - \alpha y) + cz^2(\beta x - \alpha y), \tag{2}$$

where ρ is the radial velocity of the cluster.

It is important to notice that

$$\beta x - \alpha y = \beta x_0 - \alpha y_0,$$

and that, consequently, the term ar of V, which gives the angular rotation

of the globular clusters system in the neighbourhood of O, mingles with the terms derived from the translation of the Sun :

$$\rho = -(X + ay_0)\alpha - (Y - ax_0)\beta - \gamma Z + br(\beta x - \alpha y) + cz^2(\beta x - \alpha y), \qquad (2')$$

3. One may try to determine the coefficients of (2') by means of the available radial velocities of clusters. A succession of attempts I made in that direction showed me that one cannot obtain plausible results unless one

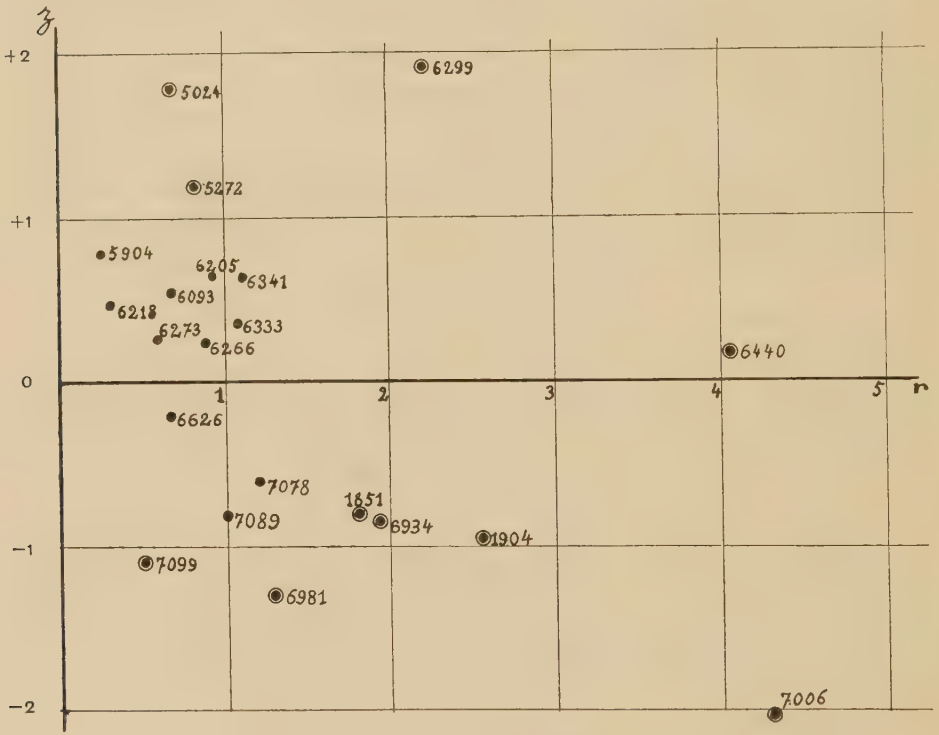

FIG. 1.—*Distribution of the r and z of the globular clusters with known radial velocities.*

The results of paragraph 3 have been obtained using exclusively the clusters situated on left of r = 2.

The results of paragraph 5 have been obtained using exclusively the circled clusters.

eliminates from the computation the clusters with too great r and z. This is quite normal since such clusters are likely to have a very slow galactic rotation, and, on the other hand, the value of formula (1) is reduced to a limited area. Therefore I have used only the 17 clusters in Table I. On fig. 1, which represents the distribution of the r and z of the clusters whose radial velocities are known, these seventeen clusters are those which correspond to $r < 2$. The radial velocities have been borrowed from a recent work of S. W. Shiveshwarkar (*M.N.*, 95, 555, 1935). For the computation of x, y, z and r we have adopted the distances of the clusters from the Sun as given by H. Shapley (*Star Clusters*, Harvard Observatory, monograph, 1930, p. 223 ; *Handbuch der astrophysik. Das Sternsystem*, erst. teil ii, ch. 5, Appendix A, p. 762).

The least squares solution applied to the seventeen clusters, equation (2′), gives

$$X + ay_0 = -173 \pm 71 \text{ km. sec.}^{-1},$$
$$Y - ax_0 = -95 \pm 110 \text{ km. sec.}^{-1},$$
$$Z = +94 \pm 33 \text{ km. sec.}^{-1},$$
$$b = +239 \pm 85 \text{ km. sec.}^{-1} (10^4 \text{ parsecs})^{-2},$$
$$c = +282 \pm 110 \text{ km. sec.}^{-1} (10^4 \text{ parsecs})^{-3}.$$

The following table gives the list of the clusters used in the computation and the residuals ρ_R :—

TABLE I

Cluster	l	b	r	z	ρ	ρ_R
N.G.C. 1,851	212°	−34°·5	1·81	−0·81	+315	− 42
5,024	305°	+79°	0·68	+1·79	−185	−167
5,272	8°	+77°·5	0·83	+1·19	−132	+ 8
5,904	332°	+46°	0·25	+0·78	+ 10	+ 21
6,093	320°·5	+18°	0·68	+0·54	+ 70	+ 9
6,205	27°	+40°	0·94	+0·66	−243	−100
6,218	344°	+25°	0·33	+0·46	+160	+113
6,266	322°	+ 7°	0·84	+0·23	+ 50	− 27
6,273	324°·5	+ 9°	0·61	+0·26	+ 30	− 43
6,333	333°	+10°	1·07	+0·36	+225	+170
6,341	36°	+35°	1·11	+0·64	−160	+ 26
6,626	335°	− 7°	0·67	−0·20	0	− 98
6,934	20°	−20°	1·94	−0·85	−350	− 53
7,078	33°	−28°	1·21	−0·61	−100	+ 4
7,089	21°	−36°	1·00	−0·82	− 10	+ 60
7,099	355°	−48°·5	0·51	−1·09	−125	−153
6,981	3°	−34°	1·29	−1·30	−180	+ 18

4. Though the mean errors are great, one can already draw some conclusions from the preceding results :

Since the angular velocity in the rotation

$$\frac{V}{r} = a + br + cz^2$$

must decrease in absolute value when the distance from O and from the xOy plane increases, it follows that the sign of both b and c must be contrary to the sign of a, which is the one of the rotation. The values found for b and c being positive, a must be negative. This is not surprising, the galactic rotation occurring in the retrograde way. It is important to notice that b and c attain values of several hundreds of km./sec. Since V/r must always be negative and since formula (2′) has been established for clusters with

$$r < 2, \qquad z < 1{\cdot}8,$$

it follows that a, too, must be considerable and attain several hundred of km./sec.

One is led to the same conclusion by studying the values of $X + ay_0$ and $Y - ax_0$, which stand for the velocity V'_0 of the Sun with regard to axes revolving around Oz with the angular velocity a. If one adopts for these quantities respectively -173 and -95 one obtains for V'_0 a velocity of 196 km./sec. directed towards longitude 209°, that is to say in the sense opposite to the rotation of the Sun.

The mean errors of $X + ay_0$ and $Y - ax_0$ being great, one can only conclude that the velocity of V'_0 is either verging to zero or directed in the sense contrary to the galactic rotation. This can only happen if a is important and greater than 300 km./sec., the velocity of the Sun in the rotation.

On the other hand, b and c being important, one can see that the value of V given by formula (1) becomes rapidly positive at a distance of 20,000 parsecs from the centre of the galaxy. Formula (1) being purely empirical and its use limited to the neighbourhood of O and of the xOy plane, it follows that the clusters whose distance from O exceeds 20,000 parsecs have a velocity of rotation negligible according to the degree of approximation we deal with here.

So we have now stated that the galactic rotation of the globular clusters attains several hundreds of km./sec. and is next to zero at 20,000 parsecs from the centre.

5. There are two ways of determining a.

One of them is to determine directly X, Y, Z by means of the very distant globular clusters which are, as we have just supposed, deprived of rotation. The equation

$$\rho = -\alpha X - \beta Y - \gamma Z$$

applied to the 10 clusters of Table II. (those clusters are represented by circles in fig. 1), gives

$$X = +279 \pm 32 \text{ km./sec.}$$
$$Y = +113 \pm 70 \text{ km./sec.}$$
$$Z = +\ 52 \pm 32 \text{ km./sec.}$$

Table II

Cluster	l	b	r	z	ρ	ρ_R
1851	212°	−34°·5	1·81	−0·81	+315	+ 41
1904	194°	−28°	2·57	−0·96	+235	− 52
5024	305°	+79°	0·68	+1·79	−185	−122
5272	8°	+77°·5	0·83	+1·19	−132	− 19
6229	40°	+40°	2·24	+1·94	−100	+153
6934	20°	−20°	1·94	−0·85	−350	− 85
7099	355°	−48°·5	0·51	−1·09	−125	+ 14
6440	335°	+ 2°	4·02	+0·17	−175	+ 31
6981	3°	−34°	1·29	−1·30	−180	+ 27
7006	32°	−21°	4·99	−2·04	−350	− 92

The equations

$$\left.\begin{aligned} X + ay_0 &= -173 \\ Y - ax_0 &= -95 \end{aligned}\right\} \qquad (3)$$

give

$$a = -500 \pm 150 \text{ km. sec.}^{-1} (10^4 \text{ parsecs})^{-1}.$$

The second way of determining a is to adopt for the velocity of the Sun in the galactic rotation the velocity

$$X = -171,$$
$$Y = -261,$$

in the direction $l = 55°$, whose magnitude (300 km./sec.) is deduced from the coefficients A and B of Oort, B being supposed exactly known :

$$\frac{V_0}{r_0} = B - A, \qquad A = +0{\cdot}015, \qquad B = -0{\cdot}015.$$

The equations (3) give

$$a = -600 \pm 150 \text{ km./sec.}$$

I adopt for a the mean of these two determinations :

$$a = -550 \pm 100 \text{ km./sec.}$$

6. It results from paragraphs 3 and 5 that the linear velocity of the globular clusters is represented in an area of 20,000 parsecs radius around the galactic centre by the empirical formula

$$V = -550r + 240r^2 + 280rz^2, \qquad (4)$$

the mean errors of the coefficient being about 100.

Let us notice that formula (4) can be applied to the stars situated in the neighbourhood of the Sun.

The quantities

$$A = -\tfrac{1}{2}\frac{V}{r} + \tfrac{1}{2}\frac{dV}{dr},$$

$$B = \tfrac{1}{2}\frac{V}{r} + \tfrac{1}{2}\frac{dV}{dr},$$

deduced from formula (4), where $r = 1$, $z = 0$, take the values

$$A = +0{\cdot}012 \text{ km. sec.}^{-1} \text{ parsec}^{-1},$$
$$B = -0{\cdot}019 \quad \text{,,} \quad \text{,,}$$

which do not differ much from Oort's values.

Let us notice too that, according to formula (4), the velocity V of a cluster situated in the neighbourhood of the Sun is -310 km./sec., which is about the velocity of rotation of the stars in our neighbourhood. This verification is not a consequence of the way of determining a by the equations (3).

Although formula (4) is thus verified one must not forget that it is no

more than an empirical and approximate formula whose single purpose is to find out the importance of V and of its variations.

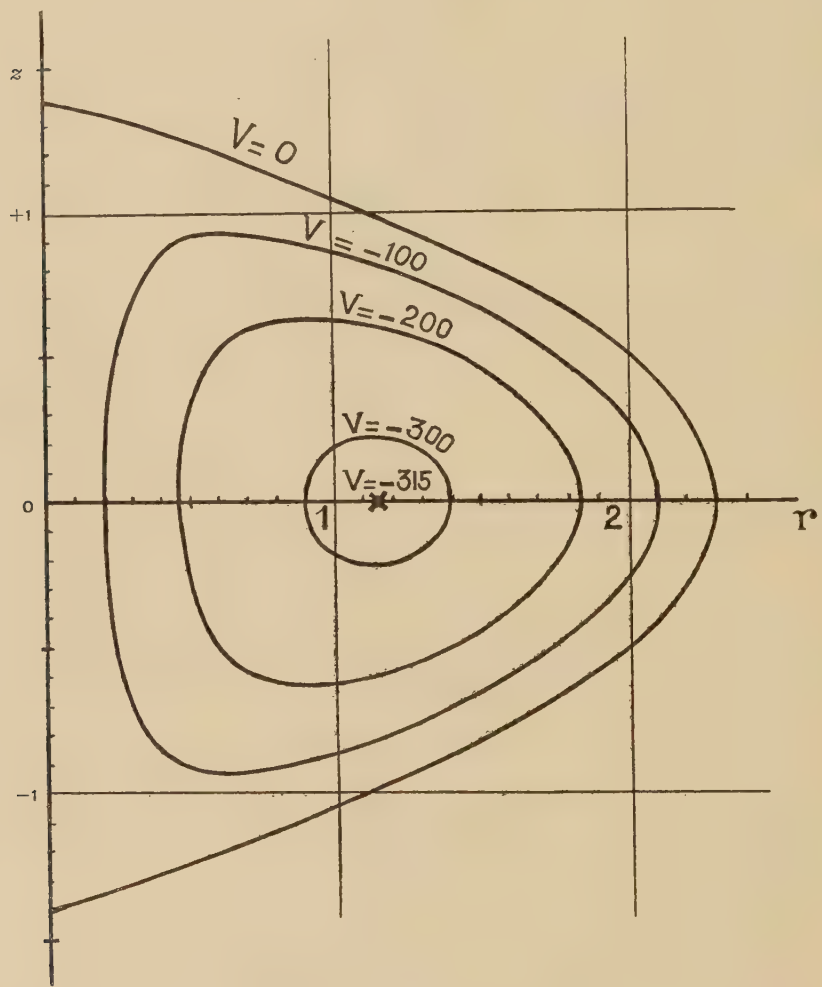

FIG. 2.—*Curves of equal linear velocities of rotation in a galactic meridian according to formula*

$$V = -550r + 240r^2 + 280z^2. \qquad (4)$$

The curves correspond to $V=0$, $V=-100$, $V=-200$, $V=-300$, *and the maximum of* V *has been indicated* ($V=-315$ *for* $r=1{\cdot}15$, $z=0$).

Fig. 2 represents some curves $V=$constant. Within the galaxy one must have

$$V < 0.$$

Fig. 2 would lead one to admit about 20,000 parsecs for the radius of the galaxy and 10,000 for its minor axis.

We see that V is a maximum near the Sun:

$$z=0, \qquad r=1{\cdot}15, \qquad V=-315.$$

Summary

The globular clusters system revolves around the axis of the galaxy with a velocity V given by formula (4).

The angular velocity of the clusters is

$$-0{\cdot}0550 \text{ km. sec.}^{-1} \text{ parsec}^{-1}$$

near the galactic centre, its absolute value decreases at increasing distances from the centre and the galactic plane, going down to zero at 20,000 parsecs from the centre in the plane and at a distance of about 10,000 parsecs from the plane.

Formula (4) fits in with the known constants of the galactic rotation of the stars around us.

I think it would be premature at the present time to indulge in dynamical considerations with reference to the above results ; such investigations must await a considerable increase in radial velocity measurements.

NOTE ON A PAPER BY MR. S. W. SHIVESHWARKAR ON THE DIRECTION OF STAR-STREAMING IN THE GALAXY.

Dr. Otto Heckmann.

(Communicated by E. A. Milne)

In *M.N.*, **95**, 655, 1935, Mr. S. W. Shiveshwarkar treats of the problem of solving the four-dimensional equation of continuity

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \dot{\varpi}\frac{\partial f}{\partial \varpi} + \dot{\theta}\frac{\partial f}{\partial \theta} + \dot{\Pi}\frac{\partial f}{\partial \Pi} + \dot{\Theta}\frac{\partial f}{\partial \Theta} = 0$$

in the case of a purely central force $K = K(\varpi)$ which demands

$$\dot{\Pi} = \frac{\Theta^2}{\varpi} - K; \quad \dot{\Theta} = -\frac{\Pi\Theta}{\varpi}.$$

He postulates the solution to be of the form

$$f = f_0 \operatorname{Exp}\left[-h^2(\Pi - \Pi_0)^2 - m(\Pi - \Pi_0)(\Theta - \Theta_0) - k^2(\Theta - \Theta_0)^2\right].$$

Further, there is assumed that f_0, Π_0, Θ_0, h^2, m, k^2 are functions of the co-ordinates alone and not of the time. From these propositions the author gets the system (12), (13), (14) of ten partial differential equations of the first order with six unknowns. It is *a priori* probable that these equations will have only very special solutions ; these solutions have escaped the author's notice. Therefore the essential points of the solutions are given in the following, omitting any peculiarities.

1. With full regard to the conditions of integrability the equations (12) have the only solution

$$h^2 = \alpha \cos 2\theta + \beta \sin 2\theta + \gamma$$
$$m = (a \cos \theta + b \sin \theta)\varpi - 2\alpha \sin 2\theta + 2\beta \cos 2\theta$$
$$k^2 = (-a \sin \theta + b \cos \theta)\varpi - \alpha \cos 2\theta - \beta \sin 2\theta + \gamma + \kappa\varpi^2,$$

where α, β, γ, a, b, κ are constants.

2. The equations (13) and the first equation (14) give

$$2h^2\Pi_0 + m\Theta_0 = 0$$
$$2k^2\Theta_0 + m\Pi_0 = \delta \cdot \varpi$$

where δ is a constant. (There is a misprint in the last equation (14) where the second Π_0 should be changed to Θ_0.)

3. The condition of integrability for the first two equations (14) gives with respect to the foregoing results

$$-\varpi[2 + F(\varpi)][a \cos \theta + b \sin \theta] + 2[F(\omega) - 1][\alpha \sin 2\theta - \beta \cos 2\theta] = 0,$$

where $F(\varpi) = d \log K / d \log \varpi$. This equation can only be fulfilled for arbitrary pairs of ϖ, θ in the following essentially different cases:—

I. $a = b = \alpha = \beta = 0$. Then and only then the central force is arbitrary. One gets

$$h^2 = \gamma\,; \quad m = 0\,; \quad k^2 = \gamma + \kappa\varpi^2\,; \quad \Pi_0 = 0\,; \quad \Theta_0 = \frac{\varpi\delta}{2(\gamma + \kappa\varpi^2)},$$

which is equivalent to the theory of pure galactic rotation in the form given by Oort (*B.A.N.*, 4, 269, 1928).

II. $a \neq 0$, b arbitrary, $\alpha = \beta = 0$. The central force becomes $K(\varpi) = \text{const.}/\varpi^2$, that means Newtonian gravitation in the field of a central body. The constant has to be taken positive in order to render the system in a physically plausible state.

III. $\alpha \neq 0$, β arbitrary, $a = b = 0$. In this case the force must be of the form $K(\varpi) = \text{const} \cdot \varpi$, that means quasi-elastic attraction if the constant is taken positive again on reasons as before.

The result is: The deviation of the vertex from the direction to the centre of the system, which Mr. Shiveshwarkar wishes to explain, is only possible in the two very special cases II and III. But if the force is arbitrary (I) there is only streaming in circular stream-lines around the centre, and one of the axes of the velocity ellipsoid is always directed radially. The present author has made some remarks on case II in *Vierteljahrsschrift der Astron. Gesellschaft*, 68, 322, 1933. A complete treatment of case II is nearly completed and will be published elsewhere.

Universitäts-Sternwarte,
Göttingen:
1935 *September* 18.

ON THE DIRECTION OF STAR-STREAMING.

Bertil Lindblad.

1. In a recently published very interesting paper * S. W. Shiveshwarkar has tried to establish a connection between a deflection of the vertex direction of star-streaming and the characteristic coefficients of differential galactic rotation and expansion.

In the numerical application of the final formula some care must be taken as to the sign of two important quantities, the angular velocity Ω of systematic rotation and the vertex deflection ψ. In the theoretical part of Shiveshwarkar's work the velocity of rotation is reckoned in direction of increasing galactic longitude, as shown by his fig. 1 and by comparison of the formulæ (18) and (19) with Oort's formulæ (1) and (2). Θ_0 is evidently used with different sign in the two sets of formulæ. In Oort's case the rotation is reckoned positive in the retrograde direction. When, on the other hand, Θ and Ω are reckoned positive in the direction of increasing λ, the vertex deviation ψ according to its definition $\tan 2\psi = \frac{m}{k^2 - h^2}$ is reckoned positive in the opposite direction, towards decreasing λ.

Using values of the coefficients A, B, C, A', valid for the B stars according to a compilation by Ogrodnikoff, and assuming $\psi = +20°$, Shiveshwarkar derives from his formula *positive* values of Ω, without noticing that the value of this quantity from the radial velocities of globular clusters should be reckoned *negative* in his system of notation.

Now $\psi = +20°$ corresponds to a vertex longitude of 305°, which is actually in the general direction of the observed vertex for the B stars,† for which the coefficients should, in fact, be valid. Still, the discordance in the sign of the computed Ω from the ordinarily accepted direction of rotation is obvious. The intention was, however, evidently to explain the deviation in the opposite direction of the more pronounced vertex of the general velocity ellipsoid of the later types. Assuming then $\psi = -20°$ we get a still larger positive value of Ω. We thus conclude that Ogrodnikoff's coefficients in no case appear to be compatible with Shiveshwarkar's theory, if we assume a retrograde motion of rotation of the galaxy. It is true that we get formal agreement with a numerically smaller negative value of ψ, but, on the other hand, this would still mean a somewhat artificial arrangement, as Ogrodnikoff's value of the coefficient A involves an expansion, and we may show that a negative ψ, *i.e.* a vertex deviation towards increasing λ, means a negative value of the mean velocity Π_0 along the radius vector. This follows from the relation

$$2h^2\Pi_0 + m\Theta_0 = 0.$$

A negative Θ_0 with negative m and ψ, will mean a negative Π_0.

* *M.N.*, **95**, 655, 1935. † Cf. *M.N.*, **90**, 503, 1930 (*Stockholm Medd.*, No. 5).

Assuming that a positive value of Π_0 means in general a differential expansion, we arrive at the result that an expansion will mean a vertex shift towards decreasing λ, and that a vertex shift towards increasing λ will in general correspond to a contraction. We shall find occasion below to confirm this rule. In the case of motions in a stationary field of force of rotational symmetry we must therefore state that the relation between a vertex deviation towards increasing λ and a general expansion as proposed by Pilowski * is not valid. It is true, however, that Pilowski introduces other theoretical conceptions, as he is inclined to consider the expansion and vertex shift as a consequence of a decreasing central force, though he makes no definite assumption on this point.

The result here derived is in qualitative agreement with the vertex deviation towards small λ and the positive K effect of the B stars. It also agrees in principle with the vertex deviation towards increasing λ and the vanishing or slightly negative K effect actually observed in the later types.

2. We note in the last results just quoted at least a tendency to an agreement between theory and observation, but it may still be worth while to enter for a moment more closely into the theoretical side of the problem. Shiveshwarkar assumes not only $\frac{Df}{Dt}=0$, but also $\frac{\partial f}{\partial t}=0$, as the time derivatives of the parameters of f are omitted. The function f is chosen to give an ellipsoidal distribution around the mean values

$$\Pi_0(\varpi,\ \theta), \qquad \Theta_0(\varpi,\ \theta).$$

It is clear, however, that if $\Pi_0 \neq 0$, and Θ_0 depends on θ, at any one point, the condition $\frac{\partial f}{\partial t}=0$ cannot at the same time be valid in the entire system. For if such should be the case, f as the solution of the equation

$$\Pi\frac{\partial f}{\partial \varpi}+\dot{\theta}\frac{\partial f}{\partial \theta}+\dot{\Pi}\frac{\partial f}{\partial \Pi}+\dot{\Theta}\frac{\partial f}{\partial \Theta}=0$$

in the supposed stationary field of force of rotational symmetry must in general be a function of the two first integrals, the energy-integral and the area-integral, and we should have identically at every point $\Pi_0=0$, $\frac{\partial \Theta_0}{\partial \theta}=0$. In other words, we may state that, with any form of f, the partial differential equations resulting from the assumption $\frac{\partial f}{\partial t}=0$ cannot in general give for the parameters of f single-valued functions of ϖ and θ with period 2π in θ, if Π_0 and $\frac{\partial \Theta_0}{\partial \theta}$ shall differ from zero at any point. Still, within a limited interval of time, brief compared with the time of revolution in θ, and for a limited region of the system, solutions for the parameters with finite Π_0 and $\frac{\partial \Theta_0}{\partial \theta}$ may be possible.

* *Z. f. Astrophysik,* **3**, 53, 279, 291, 1931.

In Shiveshwarkar's case of an *ellipsoidal* distribution with m and Π_0 differing from zero we can show, however, that the parameters cannot in general be steady functions of ϖ and θ even for a certain interval of time and within a limited region of the system. A condition for a steady ellipsoidal distribution of this kind may be drawn from Shiveshwarkar's equation (14), namely

$$\frac{\partial}{\varpi \partial \theta}(2h^2K) = \frac{\partial}{\partial \varpi}(mK).$$

From the equations (12) we derive $m = C\varpi + u(\theta)$, where C is an arbitrary constant, and $u(\theta) = \frac{\partial h^2}{\partial \theta}$. We then deduce

$$\frac{\varpi}{K}\frac{\partial K}{\partial \varpi} = \frac{2u(\theta) - C\varpi}{u(\theta) + C\varpi}.$$

As K is supposed to be a function of ϖ only, we must have either $C = 0$ or $u(\theta) = \text{const}$. We get, however, in this way quite special forms of the function K, so that the condition in question cannot be fulfilled in the general case. It is evident, however, that in such a case the strictly ellipsoidal law for the function f appears to be a too special choice of function. If we choose to neglect the time derivatives, we cannot expect the approach to an ellipsoidal distribution to be equally good at all points surrounding our Sun; in fact, the frequency function might be written more generally $f(1 + \phi)$, where the space derivatives of ϕ are perhaps not negligible, even if, with a certain approximation, ϕ can be made to vanish near the Sun. The conditions under which a *non-stationary* ellipsoidal distribution with finite Π_0, $\frac{\partial \Theta_0}{\partial \theta}$ and m within a limited region of the system is possible have been discussed very fully by Heckmann and Strassl.*

As soon as we let Π_0 and m be identically zero, we can have steady ellipsoidal distribution for all ϖ and θ, as follows directly from investigations by Eddington† and Oort.‡ In this case, however, the ellipsoidal distribution is a function of the two first integrals, as was shown for the first time by Jeans.§ We disregard here theoretical difficulties for the ellipsoidal distribution connected with the observed smallness of the axis at right angles to the galactic plane compared with the axis in the direction of the dynamical centre. These difficulties are of an essentially formal nature, because the motion at right angles to the galactic plane is dynamically practically independent of the motion in the plane (*Cf.* Oort, *loc. cit.*). We have disregarded also conditions arising from Poisson's equation, as we may consider f to define a "sub-system" of small mass compared with the total mass of the system which determines the gravitation potential V. We have reason to assume that this mass is greatly concentrated towards the central parts of the

* *Göttingen Veröffentlichungen*, Nr. 41, 1934; Nr. 43, 1935.
† *M.N.*, **76**, 37, 1915.
‡ *B.A.N.*, **4**, 159, 1928.
§ *M.N.*, **76**, 70, 1915.

system. As is well known, Eddington's work just cited discusses very fully the general conditions of a steady state with ellipsoidal velocity distribution. Oort assumes from general reasons $\Pi_0 = 0$, and then derives $m = 0$, which follows as a necessary consequence.

The points just discussed may be made clearer if we actually consider all the orbits traversing a certain point near the Sun at a certain moment. I have shown * that in a suitably chosen rotating co-ordinate system (the angular velocity of which will differ from one particle to another) the motion of a particle in the galactic plane will describe a certain closed orbit, which with decreasing size of the orbit approaches to a small ellipse, while at the same time the centre of the orbit coincides more and more closely with a point representing the circular motion which has the same area integral as the motion considered. We assume now at first matter to be distributed at random in one and the same orbit, and we consider all those orbits of given size which traverse a certain point of space to be on the average equivalent in mass. Assuming further the average amount of matter carried per unit phase-volume in orbits of various size to follow a Gaussian distribution, we then get an ellipsoidal velocity distribution, where the vertex direction in the galactic plane points exactly towards the centre of the system. The ratio of the two axes σ_1 and σ_2 in the galactic plane is $\frac{\sigma_2}{\sigma_1} = \sqrt{1 - \frac{A}{\omega}}$, where A is Oort's coefficient of differential rotation, ω the angular velocity of the circular motion. This way of reasoning, which was developed in detail by the present author in the paper just cited, seems actually to be the only general way of *deriving*, instead of simply assuming, an ellipsoidal law of stellar velocity distribution. I will call the ellipsoidal distribution of small relative velocities thus derived the "*natural*" *velocity ellipsoid.*

A deviation of the vertex direction from the direction towards the centre will necessarily mean either that different orbits of the same size passing through the point are systematically different with respect to mass, or that the distribution of matter along the relative orbits is not uniform, or distributed at random, but is at a certain moment systematically different for different parts of these orbits. For instance, the parts of the orbits with outwards directed motion may be systematically different from those parts in which matter moves inwards. It is clear, however, that if we consider a very great number of orbits such differences are likely to turn out as accidental and continually changing in character.

3. The matter stands, however, differently if we consider relative orbits of very great mass. We get here immediate connection with the theory of star-streaming (including the vertex deviation), which I have given recently.† I tried to show that in all probability the distribution of stellar velocities near the Sun is determined by the "natural" velocity ellipsoid in addition to certain streamings which follow asymptotical spiral orbits extending from the dense nuclear regions of the stellar system. The existence of such

* *Arkiv för Mat., Astr. och Fysik*, **20**A, No. 17, 1927 (*Upsala Medd.*, No. 26).
† *M.N.*, **95**, 663, 1935.

diffuse but momentarily fairly steady stream motions in our neighbourhood will to a certain extent give more of a real basis to Shiveshwarkar's method, but it will be shown presently that we get more clearly defined relations between the vertex deviation and the characteristic coefficients of differential expansion and rotation on the basis of the two-stream theory, which is, moreover, in this case the more logical one.

We consider at first a single stream with certain values of the integrals I_1 and I_2, the members of which are spread out in a vast region surrounding our Sun. (We assume these values of I_1 and I_2 to represent a point near the arrow point P_2 of the envelope curve in fig. 1 of the paper just cited.) We

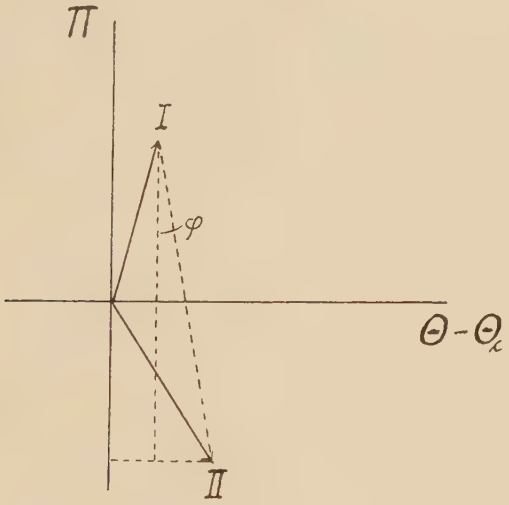

Fig. 1.

form the difference of velocity between the stream motion at the point (r, λ) and the origin, which represents the position of the Sun, assuming a stationary field of force of circular symmetry, so that the stream motion at various places is governed by the equations

$$\left.\begin{aligned} \Pi^2 + \Theta^2 - 2V &= \text{const}, \\ R\Theta &= \text{const}. \end{aligned}\right\} \tag{1}$$

(R is used instead of ϖ to denote the distance from the galactic centre.) We get for the radial and tangential components of the differential effect to the first order, λ being as before the galactic longitude reckoned from the central direction, $L = 325°$,

$$\left.\begin{aligned} W &= ra_1 + \omega r \sin 2\lambda + ra_2 \cos 2\lambda, \\ T &= \omega r \cos 2\lambda - ra_2 \sin 2\lambda, \end{aligned}\right\} \tag{2}$$

where

$$\left.\begin{aligned} a_1 &= \frac{1}{2\Pi}\left(\frac{\partial V}{\partial R} + \frac{\Theta^2}{R}\right) + \tfrac{1}{2}\frac{\Pi}{R}, \qquad \omega = \frac{\Theta}{R}, \\ a_2 &= a_1 - \frac{\Pi}{R}, \end{aligned}\right\} \tag{3}$$

Π, Θ and R in (3) being the values of these quantities at the origin. Θ and ω are reckoned positive in the retrograde direction. We now introduce another stream, forming with the first one a vertex deviation ϕ, reckoned positive in the direction of increasing λ (fig. 1). We assume, in accordance with the theory concerning the origin of the streams, that the radial components Π_{I} and Π_{II} are of nearly the same size but of opposite sign; further, that the components Θ_{I} and Θ_{II} are not far from the circular velocity Θ_c. We then have from fig. 1,

$$\tan\phi = \frac{\Theta_{\mathrm{II}} - \Theta_{\mathrm{I}}}{\Pi_{\mathrm{I}} - \Pi_{\mathrm{II}}}. \tag{4}$$

For the differential effects we get expressions analogous to (2), where α_1 and α_2 will be replaced by the same mean $\bar{\alpha}$ for the two streams. We can write

$$\left.\begin{aligned} \alpha_1 &= \frac{1}{2\Pi R}(\Theta^2 - \Theta_c^2) + \tfrac{1}{2}\frac{\Pi}{R} = \frac{\omega}{\Pi}(\Theta - \Theta_c) + \tfrac{1}{2}\frac{\Pi}{R}, \\ \alpha_2 &= \frac{\omega}{\Pi}(\Theta - \Theta_c) - \tfrac{1}{2}\frac{\Pi}{R}, \end{aligned}\right\} \tag{5}$$

and then

$$\bar{\alpha} = \bar{\alpha}_1 = \bar{\alpha}_2 = \frac{\omega}{2\,|\,\Pi\,|}(\Theta_{\mathrm{I}} - \Theta_{\mathrm{II}}) = -\omega\tan\phi. \tag{6}$$

We then get easily

$$\left.\begin{aligned} W &= r[-\omega\tan\phi + A_0\sin 2(\lambda - \tfrac{1}{2}\phi)], \\ T &= rA_0\cos 2(\lambda - \tfrac{1}{2}\phi), \end{aligned}\right\} \tag{7}$$

where

$$A_0 = \omega\sqrt{1 + \tan^2\phi}. \tag{8}$$

We have thus obtained a very simple expression for the differential effects in terms of the vertex deviation ϕ. We observe that a vertex deviation in the direction of increasing λ gives a slight contraction, $r\omega\tan\phi$, of the material in question. [$(\Pi_0 = \frac{1}{2}(\Pi_{\mathrm{I}} + \Pi_{\mathrm{II}})$] is assumed to be small, but its sign is not determined.) The amplitude A_0 comes out much larger than in the ordinary differential rotation; in fact, it will obviously exceed ω, whereas it ordinarily is of the order $\frac{1}{2}\omega$. The apparent longitude of the centre of the system is shifted from the true direction towards the centre an angle equal to half the vertex deviation. There is no constant term in the tangential effect T.

Assuming a mixture of the matter of the two streams with that of the "natural" ellipsoid with normal differential rotation of coefficients A and B in Oort's system of notation, we obtain, if the streams enter in the proportion ϵ, and the amount of "normal" matter is $1 - \epsilon$,

$$\left.\begin{aligned} W &= r[-A_0\sin 2\lambda_0 + A_0\sin 2(\lambda - \lambda_0)], \\ T &= r[B_0 + A_0\cos 2(\lambda - \lambda_0)], \end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned} A_0 &= \{(1-\epsilon)A + \epsilon\omega\}\sqrt{1+\tan^2 2\lambda_0}\,; \qquad B_0 = (1-\epsilon)B \\ \tan 2\lambda_0 &= \frac{\omega \tan \phi'}{(1-\epsilon)A + \epsilon\omega}, \qquad \tan \phi' = \epsilon \tan \phi. \end{aligned}\right\} \tag{10}$$

ϕ' may be identified approximately with the "effective" vertex deviation.

The evidence from observed radial velocities * seems to corroborate the theoretical results obtained here, giving for the ordinary stars of types A–K an indication of a positive λ_0 and a large coefficient A_0. As mentioned before, a slightly negative K-effect is indicated for these types. Further, there is in Oort's results an indication that the effective coefficient B_0 is numerically larger for those groups which do not partake perceptibly in the general star-streaming (B-, c-, O-, N-, δ Cephei-stars) than for the stars in general. It is possible that the latter value leads to an underestimate of ω when used for B in the relation $\omega = A - B$. On the other hand, the value of B_0 derived by Plaskett and Pearce † for the O–B stars agrees nearly with that for the stars in general, so that the matter is rather intricate, and cannot be settled at present.—The most striking confirmation is, however, obtained from the proper motions, when the differential rotation is determined from the second harmonic terms in α and δ. These effects, which were already discovered by Gyldén,‡ have been studied especially by Schilt,§ J. E. Merrill ‖ and S. Holm.¶ The mean results from Gyllenberg's zone (35°–40°),** Prager's zone (31°–40°), the Yale zone (50°–55°) †† and Boss PGC ‡‡ are

$$A_0 = 0''{\cdot}0098, \qquad L_0 = 349°, \qquad \lambda_0 = +24°.$$

Assuming $\omega = 0''{\cdot}0062$, $A = 0''{\cdot}0033$, $\phi' = +20°$, we should expect theoretically, if $\epsilon = 0{\cdot}5$ in our neighbourhood,

$$A_0 = 0''{\cdot}0053, \qquad L_0 = 338°, \qquad \lambda_0 = +13°.$$

We might mention that the value of ω is rather on the conservative side and may actually be higher. With Oort's value of B from the distant groups, $B = 0''{\cdot}0050$, we should have $\omega = 0''{\cdot}0083$, $A_0 = 0''{\cdot}0065$. The

* *Cf.* Oort, *B.A.N.*, **4**, 79, 1927; Lindblad, *M.N.*, **90**, 503, 1930 (*Stockholm Medd.*, No. 5).

† *M.N.*, **94**, 679, 1934.

‡ *Öfversikt av Kungl. Vetenskapsakademiens Förhandlingar, Stockholm*, **28**, 956, 1871.

§ *Astr. J.*, **39**, 17, 1928.

‖ *Astr. J.*, **39**, 90, 1928.

¶ *Lund Medd.*, Ser. II., Nr. 55, 1930.

** Holm's results, mean for different magnitude groups.

†† Mean of two results by slightly different methods.

‡‡ It is remarkable that the Boss catalogue, when the material is treated in this way, gives higher values of A_0 and L_0 than when the components of proper motion in galactic longitude are formed and studied. It seems possible, however, as suggested by Schilt, that the systematic errors of the proper motions may have a more dangerous influence in the latter case, and more or less tend to flatten the double wave of the differential effect.

agreement is remarkably good with the results from Prager's zone, which are

$$A_0 = 0''.0081, \qquad L_0 = 337^\circ, \qquad \lambda_0 = +12^\circ.$$

It is necessary to mention at last the obvious conclusion, that if Θ_I and Θ_{II} differ slightly from Θ_c, the presence of the streams will affect to some extent the observed solar motion of the group of stars in question. We get here a basis for a discussion of the observed difference in solar velocity between the B stars on one hand, the later types with pronounced star-streaming and direct vertex deviation on the other. It is possible that the centroid of the B stars, which gives a slightly higher solar velocity, represents the circular motion Θ_c more closely than the mean of the stars of somewhat later types. The Θ values of streams I and II perhaps slightly exceed Θ_c, *i.e.* the streams overrun the circular motion, a phenomenon quite within the possibilities of the present theory. In this connection we must, however, consider another important point. The assumption of constant I_1 and I_2 for each one of the two streams is of course a schematical assumption only. There is a certain dispersion here for each one of the streams, and theoretically as well as empirically there is a continuous transition from the two streams to the showers of high velocity stars which occur in nearly the same directions. In Redman's * material of K stars the differential effects are probably much affected by stars on the verge between the streams I and II and the high velocity groups. In H. Nordström's † discussion of this material the influence of these stars has been reduced. The high velocity stars possess, however, on the average an asymmetrical drift, *i.e.* a smaller mean value of Θ_0, which will make the observed solar motion increase. Consequently, in material containing a large proportion of high velocity stars the solar motion will at last even surpass that of the B stars.

The results from the B stars indicating a fairly small A_0, positive K, slightly negative λ_0, and a large vertex deviation towards smaller galactic longitude, are in qualitative agreement with our theory. On account of the small dispersion in the "natural" ellipsoid in this class, we may assume ϕ to represent the effective vertex direction more closely than ϕ', even if we assume ϵ to be a small quantity. Even with a numerically large ϕ we may then expect λ_0 to be numerically small and A_0 to be close to A. Our present stream theory cannot be strictly valid here, but may still contain some approximation to the truth, as we know that there are several indications of vast individual streams among these stars. We may assume that the combination of I_1 and I_2 for the streams in our neighbourhood may be arranged naturally into two groups, one of positive, the other of negative Π. It is possible, however, that the mean motion Π_0 of the two streams thus formed will differ considerably from zero. In that case the formula (6) for the expansion coefficient $\bar{a}_1$ may be written more correctly

$$\bar{a}_1 = \frac{\omega(\Pi_{II} - \Pi_I)}{4\Pi_I\Pi_{II}}(\Theta_I - \Theta_{II}) + \frac{\omega\Pi_0}{\Pi_I\Pi_{II}}(\Theta_0 - \Theta_c) + \frac{\Pi_0}{2R} = -\omega \tan\phi + \frac{\omega\Pi_0}{\Pi_I\Pi_{II}}(\Theta_0 - \Theta_c) + \frac{\Pi_0}{2R}. \qquad (11)$$

* *M.N.*, **92**, 107, 1931. † *Lund Medd.*, No. 131, 1933.

The first term is positive for a negative vertex shift in galactic longitude. Π_I and Π_{II} are supposed to be of opposite sign, and Θ_0 should ordinarily be expected to lie somewhat below Θ_c for a sub-system in the galaxy like that we may assume the B stars to form. Then we see that the second term should be positive for positive Π_0. In the present theory, however, there is no direct relation between Π_0 and the sign of ϕ. The term $-r\omega \tan \phi$ seems sufficient to explain the main part of the observed K term of the apparently bright B stars, and it appears to be uncertain at present if the mean radial velocity component Π_0 actually differs appreciably from zero.

Summary

It is shown that the coefficients of differential galactic rotation and expansion given by Ogrodnikoff for the B stars are not in numerical agreement with Shiveshwarkar's theory of the vertex deviation of star-streaming. A vertex deviation in the direction of increasing galactic longitude demands a slight *contraction* of the material in question.

A theory of the vertex deviation in principle similar to that of Shiveshwarkar, but starting from the theory proposed by the present author in which the velocity distribution is considered as a combination of the "natural" velocity ellipsoid with streams of matter emanating in asymptotic spiral orbits from the dense nuclear regions of our system, is developed. This leads to particularly simple formulæ for the differential effects, which stand in good agreement with observation, especially with the second harmonic terms in the proper motions in α and δ. The probable significance of the difference in state of motion between the B-type stars and the stars of later types is discussed.

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ABSORPTION COEFFICIENTS AND HYDROGEN LINE INTENSITIES.

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(Communicated by Harlow Shapley)

1. *General Formulæ for Transition Probabilities in a Coulomb Field between States of Positive or Negative Energy*

The problem of the absorption of light by electrons in the neighbourhood of an ion has been discussed by many authors in connection with the theory of the stellar absorption coefficient. The general expressions for the absorption coefficient obtained from the wave-mechanical treatment of the problem are in the form of double series the summation of which has been carried out only in a number of limiting cases. Closed expressions for the intensities of lines in the Lyman and Balmer series were first obtained by

Pauli * and later checked by Sugiura,† who extended the formulæ to the Paschen series and included the case of the bound-free transitions. The original work of Kramers,‡ which was based on the classical theory of light and dealt with intensities arising from bound-free and free-free transitions, was shown by Gaunt § to be valid in the region of large quantum numbers. Oppenheimer ‖ has given a general solution of the problem, though his formulæ are not reduced to the simplest form and his asymptotic approximations are in error. Maue ¶ has further extended the results of Gaunt.

Recently, L. McLean ** has derived a closed expression for the intensities in the case of transitions between discrete states. We give, without formal proof, a somewhat more general formula than that of McLean, which applies to the bound-free and free-free transitions as well, and have derived convenient asymptotic approximations for these formulæ.

The frequency of a line in a hydrogenic atom is given by the usual formula:

$$\nu_{nn'} = R\left(\frac{1}{n'^2} - \frac{1}{n^2}\right), \tag{1.1}$$

where n' and n are the respective quantum numbers of the lower and upper states. This expression may be extended to include the bound-free and free-free transitions if we let $n \to i\kappa$ and $n' \to il$, where κ and l are real positive numbers lying in the range 0 to ∞. They are not necessarily integers. When we adopt this convention, the distinction between an ionized and a neutral atom is only formal. We may regard an ionized atom as a neutral atom with its electron in a hyperbolic instead of in an elliptic orbit.

The total energy emitted per cm.^{3} per second by an assembly containing N_n atoms per cm.^{3} in state n is, by definition,

$$E_\nu = N_n A_{nn'} h\nu_{nn'}, \tag{1.2}$$

where $A_{nn'}$ is the Einstein probability coefficient of spontaneous transition. By Ladenburg's relation

$$A_{nn'} = \frac{\varpi_{n'}}{\varpi_n} \frac{8\pi^2 \epsilon^2 \nu^2_{nn'}}{mc^3} f_{n,n'}, \tag{1.3}$$

where $\varpi_{n'}$ and ϖ_n are the statistical weights of the respective states and $f_{n,n'}$ is the Ladenburg f, sometimes called the oscillator strength or number of dispersion electrons per atom. It does not appear to have been realized that equation (1.3) is applicable to transitions from continuous as well as from discrete states, when the weight factor is properly defined.

* *Cf.* Schroedinger, *Ann. der Phys.*, **80**, 437, 1926.

† Sugiura, *Jour. de Phys.*, **8**, 113, 1927; *Sci. Pap. Inst. Phys. and Chem. Research (Tokyo)*, **11**, 1, 1929.

‡ Kramers, *Phil. Mag.*, **44**, 836, 1923.

§ Gaunt, *Phil. Trans.*, A, **229**, 163, 1930.

‖ Oppenheimer, *Zs. für Phys.*, **41**, 768, 1927; *ibid.*, **55**, 725, 1929.

¶ Maue, *Ann. der Phys.*, **13**, 161, 1932.

** McLean, *Nature*, **124**, 25, 1932. Also, *Phil. Mag.* (7), **18**, 845, 1934.

For a discrete state of hydrogen

$$\varpi_n = 2n^2. \tag{1.4}$$

The factor 2 arises from electron spin, for which we shall sometimes write ϖ_e. For a continuous state the choice of weight factor depends upon the definition of N_n, the number of atoms in a continuous state. Several possible choices are available. We may, for example, define N_n as the number of atoms per cm.^{3} whose electrons in hyperbolic orbits have velocities at infinity ranging from v to $v+dv$. As Fowler * has shown, the proper weighting in this region of phase space, where the distribution is essentially classical, is obtained by assigning a weight ϖ' to the cell characterized by component momenta lying between p_1 and p_1+dp_1, p_2 and p_2+dp_2, p_3 and p_3+dp_3, and located in the volume element $dq_1dq_2dq_3$, where

$$\varpi' = \varpi_e \frac{dp_1dp_2dp_3dq_1dq_2dq_3}{h^3}. \tag{1.5}$$

Expressing the component momenta in terms of the actual velocity v of the electron, we find, on introducing polar co-ordinates,

$$\varpi' = \varpi_e \frac{mdv \,.\, mvd\theta \,.\, mv \sin\theta d\phi dq_1dq_2dq_3}{h^3}. \tag{1.6}$$

To obtain the total weight for the electronic state corresponding to the velocity range v to $v+dv$, we integrate this expression over the purely geometrical variables ϕ and θ, respectively from 0 to 2π and from 0 to π, and likewise over the average volume occupied by an electron. If there are N_e electrons per cm.^{3}, then this volume is clearly $1/N_e$. Thus, we obtain finally

$$\varpi_n = \varpi_e \frac{4\pi m^3v^2dv}{N_eh^3}, \tag{1.7}$$

which is probably the most convenient form. In order to compare certain of our results with those of Gaunt and Maue, however, we may define N_n as the number of electrons per unit of frequency range, and also refer the results to unit electron density. By the equations of the photoelectric effect,

$$h(\nu - \nu_0) = \tfrac{1}{2}mv^2 \tag{1.8}$$

and

$$hdv = mvdv, \tag{1.9}$$

which, combined with (1.7), give

$$\varpi_n'' = \varpi_e \frac{4\pi m^2v}{h^2}, \tag{1.10}$$

on setting N_e and dv equal to unity.

In thermodynamic equilibrium the absorption and emission must be equal. We have

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* Fowler, *Statistical Mechanics*, p. 14 (Cambridge University Press, 1929).

$$4\pi N_{n'}\int_0^\infty \alpha_{nn'}(\nu)I_\nu(1-e^{-h\nu/kT})d\nu = 4\pi N_{n'}I_\nu(1-e^{-h\nu/kT})\int_0^\infty \alpha_{nn'}(\nu)d\nu = E_\nu, \quad (1.11)$$

where $\alpha_{nn'}(\nu)$ is the absorption coefficient per atom in state n', and I_ν is given by Planck's formula

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{(e^{h\nu/kT}-1)}. \quad (1.12)$$

The factors I_ν and $(1-e^{-h\nu/kT})$ have been removed from the parentheses in the customary manner, on the assumption that the emission line is narrow so that ν does not change appreciably over the effective range of integration. The factor in parentheses takes care of the "stimulated emissions," which should always be treated as if they were negative absorptions, since they act to decrease the opacity. With weights defined as above, Boltzmann's formula is generally applicable to discrete and continuous states :

$$\frac{N_n}{N_{n'}} = \frac{\varpi_n}{\varpi_{n'}}e^{-h\nu_{nn'}/kT}. \quad (1.13)$$

When (1.7) is substituted in (1.13) and the resulted expression integrated over the velocity range from 0 to ∞, the ordinary dissociation formula is immediately obtained.* These equations are easily seen to be consistent with the definition of the integrated absorption coefficient

$$\int_0^\infty \alpha_{nn'}(\nu)d\nu = \frac{\pi\epsilon^2}{mc}f_{nn'}. \quad (1.14)$$

Our general formula for the oscillator strength is as follows :—

$$f_{nn'} = \frac{2^6}{3}\frac{D}{\varpi_{n'}}\left|\frac{[(n-n')/(n+n')]^{2n+2n'}}{n^2n'^2\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^3}\frac{\Delta(n,n')}{(n-n')}\right|. \quad (1.15)$$

$$\Delta \equiv \left[F\left(-n+1,\ -n',\ 1,\ -\frac{4nn'}{(n-n')^2}\right)\right]^2 - \left[F\left(-n'+1,\ -n,\ 1,\ -\frac{4nn'}{(n-n')^2}\right)\right]^2, \quad (1.16)$$

where F is the ordinary hypergeometric function :

$$F(\alpha,\beta,\gamma,\chi) = 1+\frac{\alpha\beta}{1!\gamma}\chi+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}\chi^2+\ \ldots \quad (1.17)$$

D is the normalizing factor. For transitions between discrete states,

$$D_{\mathrm{I}} = 1. \quad (1.18)$$

For bound-free transitions,

$$D_{\mathrm{II}} = (1-e^{-2\pi\kappa})^{-1}; \quad (1.19)$$

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* *Cf.* Menzel, *Proc. N.A.S.*, **19**, 40, 1933.

while for free-free transitions,*

$$D_{\text{III}} = (1 - e^{-2\pi\kappa})^{-1}(1 - e^{-2\pi l})^{-1} e^{2\pi l}. \tag{1.20}$$

The vertical bars signify, as usual, that the absolute value is to be taken of the complex expression that results when $i\kappa$ or il is substituted for n or n'. When n and n' are both real no difficulty arises, and McLean's formula is immediately obtained. When $n \equiv i\kappa$, (1.15) becomes

$$f_{\kappa n'} = \frac{2^5}{3}\frac{1}{n'^2}\frac{e^{-4\kappa \tan^{-1}(n'/\kappa)}}{\kappa^3 n'^3\left(\frac{1}{n'^2}+\frac{1}{\kappa^2}\right)^{7/2}(1-e^{-2\pi\kappa})}\,|\,\Delta(i\kappa, n')|, \tag{1.21}$$

for bound-free transitions. For the free-free case,

$$f_{\kappa l} = \frac{2^6}{3}\frac{1}{\varpi_l}\frac{e^{-2\pi\kappa}\,|\,\Delta(i\kappa, il)\,|}{\kappa^2 l^2\left(\frac{1}{\kappa^2}-\frac{1}{l^2}\right)^3(l-\kappa)(1-e^{-2\pi\kappa})(1-e^{-2\pi l})}, \tag{1.22}$$

where ϖ_l is given by (1.7) or (1.10).

The problem of calculating the various Δ's remains. As long as n' is real, the various hypergeometric functions degenerate to polynomials, that become increasingly complicated as n' increases. We have computed the Δ's for a number of cases and have found that they may be factorized as follows :—

$$\Delta(n, 1) = 8n(n+1)/(n-1)^2, \tag{1.23}$$

$$\Delta(n, 2) = [16n(n+2)/(n-2)^6](3n^2-4)(5n^2-4), \tag{1.24}$$

$$\Delta(n, 3) = [24n(n+3)/(n-3)^{10}](13n^4-78n^2+81)(29n^4-126n^2+81), \tag{1.25}$$

$$\Delta(n, 4) = [32n(n+4)/9(n-4)^{14}](197n^6-3152n^4+13056n^2-12288)\cdot \cdot(539n^6-6800n^4+20736n^2-12288), \tag{1.26}$$

$$\Delta(n, 5) = [40n(n+5)/9(n-5)^{18}](1083n^8-36100n^6+372250n^4-1312500n^2+ +1171875)\cdot \cdot(3467n^8-95700n^6+786250n^4-2062500n^2+1171875), \tag{1.27}$$

$$\begin{aligned}\Delta(n, 7) = [56n(n+7)/2025(n-7)^{26}](567409n^{12}-5606082n^{10}+ \\ +1993432651n^8-32628840888n^6+2428998901 35n^4- \\ -737260399890n^2+622857924045)\cdot \\ \cdot(2297425n^{12}-196890722n^{10}+6067218955n^8- \\ -829016031 48n^6+50024060677 5n^4-1144024758450n^2+ \\ +622857924045).\end{aligned} \tag{1.28}$$

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* This is the appropriate factor if we write

$$\left(\frac{n-n'}{n+n'}\right) = e^{i\pi}\left(\frac{l-\kappa}{l+\kappa}\right),$$

so that

$$\left|\left[\frac{(n-n')}{(n+n')}\right]^{2n+2n'}\right| = e^{-2\pi\kappa-2\pi l}.$$

These expressions are valid also for $n = i\kappa$, and hence may be used for obtaining $|\Delta(i\kappa, n')|$. In this case the bracketed factors in the foregoing equations, aside from numerical constants, reduce to the form

$$\left|\frac{i\kappa(i\kappa+n')}{(i\kappa-n')^{4n'-2}}\right| = \frac{\kappa}{(\kappa^2+n'^2)^{2n'-3/2}}. \tag{1.29}$$

The factors in parentheses contain only even powers of $i\kappa$ and are, therefore, real. The signs, however, all become positive. Thus

$$\left|\Delta(i\kappa, 2)\right| = \frac{16\kappa}{(\kappa^2+4)^{5/2}}(3n^2+4)(5n^2+4), \tag{1.30}$$

and similarly for other values of n'. The resulting expressions agree with those of Sugiura * for the discrete-discrete and bound-free cases, where $n' = 1$, 2 and 3, as given in his equations (79), (84), (94) and (80), (88), (100) respectively. Values of $f_{n,n'}$, computed from the exact equation (1.15) for all possible discrete transitions up to and including $n = 14$, are given in Tables Ia and Ib. The values for the Lyman, Balmer and Paschen series have been extended to $n = 35$. The corresponding Einstein A's, from (1.3), are given in Tables IIa and IIb. These values agree closely with those given by Slack † up to and including $n = 7$.

TABLE Ia

Values of $f_{n,n'}$ computed from equation (1.15)

n \ n'	1	2	3	n \ n'	1	2	3
2	$4{\cdot}162 \times 10^{-1}$			19	$2{\cdot}295 \times 10^{-4}$	$5{\cdot}167 \times 10^{-4}$	$8{\cdot}364 \times 10^{-4}$
3	$7{\cdot}910 \times 10^{-2}$	$6{\cdot}408 \times 10^{-1}$		20	$1{\cdot}966 \times 10^{-4}$	$4{\cdot}418 \times 10^{-4}$	$7{\cdot}117 \times 10^{-4}$
4	$2{\cdot}899 \times 10^{-2}$	$1{\cdot}193 \times 10^{-1}$	$8{\cdot}420 \times 10^{-1}$	21	$1{\cdot}698 \times 10^{-4}$	$3{\cdot}803 \times 10^{-4}$	$6{\cdot}111 \times 10^{-4}$
5	$1{\cdot}394 \times 10^{-2}$	$4{\cdot}467 \times 10^{-2}$	$1{\cdot}506 \times 10^{-1}$	22	$1{\cdot}476 \times 10^{-4}$	$3{\cdot}302 \times 10^{-4}$	$5{\cdot}286 \times 10^{-4}$
6	$7{\cdot}800 \times 10^{-3}$	$2{\cdot}209 \times 10^{-2}$	$5{\cdot}585 \times 10^{-2}$	23	$1{\cdot}276 \times 10^{-4}$	$2{\cdot}885 \times 10^{-4}$	$4{\cdot}608 \times 10^{-4}$
7	$4{\cdot}814 \times 10^{-3}$	$1{\cdot}271 \times 10^{-2}$	$2{\cdot}768 \times 10^{-2}$	24	$1{\cdot}137 \times 10^{-4}$	$2{\cdot}534 \times 10^{-4}$	$4{\cdot}040 \times 10^{-4}$
8	$3{\cdot}184 \times 10^{-3}$	$8{\cdot}037 \times 10^{-3}$	$1{\cdot}604 \times 10^{-2}$	25	$1{\cdot}005 \times 10^{-4}$	$2{\cdot}240 \times 10^{-4}$	$3{\cdot}558 \times 10^{-4}$
9	$2{\cdot}216 \times 10^{-3}$	$5{\cdot}429 \times 10^{-3}$	$1{\cdot}023 \times 10^{-2}$	26	$8{\cdot}931 \times 10^{-5}$	$1{\cdot}987 \times 10^{-4}$	$3{\cdot}155 \times 10^{-4}$
10	$1{\cdot}605 \times 10^{-3}$	$3{\cdot}851 \times 10^{-3}$	$6{\cdot}981 \times 10^{-3}$	27	$7{\cdot}963 \times 10^{-5}$	$1{\cdot}772 \times 10^{-4}$	$2{\cdot}809 \times 10^{-4}$
11	$1{\cdot}201 \times 10^{-3}$	$2{\cdot}836 \times 10^{-3}$	$4{\cdot}996 \times 10^{-3}$	28	$7{\cdot}138 \times 10^{-5}$	$1{\cdot}587 \times 10^{-4}$	$2{\cdot}513 \times 10^{-4}$
12	$9{\cdot}215 \times 10^{-4}$	$2{\cdot}150 \times 10^{-3}$	$3{\cdot}711 \times 10^{-3}$	29	$6{\cdot}431 \times 10^{-5}$	$1{\cdot}427 \times 10^{-4}$	$2{\cdot}243 \times 10^{-4}$
13	$7{\cdot}226 \times 10^{-4}$	$1{\cdot}672 \times 10^{-3}$	$2{\cdot}839 \times 10^{-3}$	30	$5{\cdot}809 \times 10^{-5}$	$1{\cdot}288 \times 10^{-4}$	$2{\cdot}034 \times 10^{-4}$
14	$5{\cdot}774 \times 10^{-4}$	$1{\cdot}326 \times 10^{-3}$	$2{\cdot}223 \times 10^{-3}$	31	$5{\cdot}260 \times 10^{-5}$	$1{\cdot}167 \times 10^{-4}$	$1{\cdot}840 \times 10^{-4}$
15	$4{\cdot}687 \times 10^{-4}$	$1{\cdot}070 \times 10^{-3}$	$1{\cdot}776 \times 10^{-3}$	32	$4{\cdot}784 \times 10^{-5}$	$1{\cdot}060 \times 10^{-4}$	$1{\cdot}670 \times 10^{-4}$
16	$3{\cdot}855 \times 10^{-4}$	$8{\cdot}770 \times 10^{-4}$	$1{\cdot}443 \times 10^{-3}$	33	$4{\cdot}359 \times 10^{-5}$	$9{\cdot}654 \times 10^{-5}$	$1{\cdot}521 \times 10^{-4}$
17	$3{\cdot}211 \times 10^{-4}$	$7{\cdot}273 \times 10^{-4}$	$1{\cdot}189 \times 10^{-3}$	34	$3{\cdot}982 \times 10^{-5}$	$8{\cdot}829 \times 10^{-5}$	$1{\cdot}389 \times 10^{-4}$
18	$2{\cdot}703 \times 10^{-4}$	$6{\cdot}098 \times 10^{-4}$	$9{\cdot}914 \times 10^{-4}$	35	$3{\cdot}656 \times 10^{-5}$	$8{\cdot}084 \times 10^{-5}$	$1{\cdot}272 \times 10^{-4}$

* Sugiura, *op. cit.*

† Slack, *Phys. Rev.*, **31**, 527, 1928.

TABLE Ib

Values of $f_{nn'}$ computed from equation (1.15)

n' \ n	5	6	7	8	9
4	1.038×10^{0}	1.794×10^{-1}	6.551×10^{-2}	3.229×10^{-2}	1.872×10^{-2}
5		1.231×10^{0}	2.070×10^{-1}	7.455×10^{-2}	3.644×10^{-2}
6			1.425×10^{0}	2.340×10^{-1}	8.318×10^{-2}
7				1.615×10^{0}	2.595×10^{-1}
8					1.810×10^{0}
9					
10					
11					
12					
13					

n' \ n	10	11	12	13	14
4	1.195×10^{-2}	8.185×10^{-3}	5.888×10^{-3}	4.394×10^{-3}	3.375×10^{-3}
5	2.102×10^{-2}	1.345×10^{-2}	9.208×10^{-3}	6.630×10^{-3}	4.959×10^{-3}
6	4.035×10^{-2}	2.320×10^{-2}	1.479×10^{-2}	1.013×10^{-2}	7.299×10^{-3}
7	9.162×10^{-2}	4.415×10^{-2}	2.525×10^{-2}	1.605×10^{-2}	1.096×10^{-2}
8	2.876×10^{-1}	1.001×10^{-1}	4.792×10^{-2}	2.725×10^{-2}	1.733×10^{-2}
9	2.001×10^{0}	3.140×10^{-1}	1.084×10^{-1}	5.147×10^{-2}	2.915×10^{-2}
10		2.191×10^{0}	3.410×10^{-1}	1.180×10^{-1}	5.514×10^{-2}
11			2.380×10^{0}	3.658×10^{-1}	1.245×10^{-1}
12				2.572×10^{0}	3.928×10^{-1}
13					2.764×10^{0}

TABLE IIa

Values of $A_{nn'}$ computed from equation (1.3)

n \ n'	1	2	3	n \ n'	1	2	3
2	4.67×10^{8}			19	5.05×10^{3}	2.79×10^{3}	1.95×10^{3}
3	5.54×10^{7}	4.39×10^{7}		20	3.91×10^{3}	2.16×10^{3}	1.51×10^{3}
4	1.27×10^{7}	8.37×10^{6}	8.94×10^{6}	21	3.06×10^{3}	1.69×10^{3}	1.18×10^{3}
5	4.10×10^{6}	2.52×10^{6}	2.19×10^{6}	22	2.42×10^{3}	1.34×10^{3}	9.33×10^{2}
6	1.64×10^{6}	9.68×10^{5}	7.74×10^{5}	23	1.92×10^{3}	1.07×10^{3}	7.47×10^{2}
7	7.53×10^{5}	4.37×10^{5}	3.34×10^{5}	24	1.57×10^{3}	8.66×10^{2}	6.03×10^{2}
8	3.85×10^{5}	2.20×10^{5}	1.64×10^{5}	25	1.28×10^{3}	7.06×10^{2}	4.91×10^{2}
9	2.13×10^{5}	1.21×10^{5}	8.85×10^{4}	26	1.05×10^{3}	5.80×10^{2}	4.07×10^{2}
10	1.26×10^{5}	7.08×10^{4}	5.13×10^{4}	27	8.70×10^{2}	4.80×10^{2}	3.33×10^{2}
11	7.79×10^{4}	4.37×10^{4}	3.14×10^{4}	28	7.25×10^{2}	4.00×10^{2}	2.78×10^{2}
12	5.04×10^{4}	2.82×10^{4}	2.01×10^{4}	29	6.09×10^{2}	3.35×10^{2}	2.32×10^{2}
13	3.37×10^{4}	1.88×10^{4}	1.34×10^{4}	30	5.14×10^{2}	2.83×10^{2}	1.96×10^{2}
14	2.33×10^{4}	1.30×10^{4}	9.16×10^{3}	31	4.36×10^{2}	2.40×10^{2}	1.67×10^{2}
15	1.65×10^{4}	9.16×10^{3}	6.45×10^{3}	32	3.72×10^{2}	2.05×10^{2}	1.42×10^{2}
16	1.19×10^{4}	6.63×10^{3}	4.65×10^{3}	33	3.19×10^{2}	1.76×10^{2}	1.22×10^{2}
17	8.81×10^{3}	4.89×10^{3}	3.43×10^{3}	34	2.74×10^{2}	1.51×10^{2}	1.05×10^{2}
18	6.62×10^{3}	3.67×10^{3}	2.57×10^{3}	35	2.38×10^{2}	1.31×10^{2}	9.08×10^{1}

TABLE IIB

Values of $A_{nn'}$ computed from equation (1.3)

n / n'	5	6	7	8	9
4	2.68×10^6	7.67×10^5	3.03×10^5	1.42×10^5	7.43×10^4
5		1.02×10^6	3.24×10^5	1.38×10^5	6.87×10^4
6			4.50×10^5	1.55×10^5	7.03×10^4
7				2.26×10^5	8.15×10^4
8					1.23×10^5
9					
10					
11					
12					
13					

n / n'	10	11	12	13	14
4	4.21×10^4	2.54×10^4	1.61×10^4	1.06×10^4	7.25×10^3
5	3.78×10^4	2.23×10^4	1.39×10^4	9.10×10^3	6.15×10^3
6	3.67×10^4	2.10×10^4	1.28×10^4	8.23×10^3	5.50×10^3
7	3.88×10^4	2.11×10^4	1.24×10^4	7.80×10^3	5.12×10^3
8	4.65×10^4	2.29×10^4	1.28×10^4	7.77×10^3	5.00×10^3
9	7.12×10^4	2.80×10^4	1.42×10^4	8.14×10^3	5.05×10^3
10		4.35×10^4	1.77×10^4	9.29×10^3	5.39×10^3
11			2.78×10^4	1.15×10^4	6.14×10^3
12				1.85×10^4	7.82×10^3
13					1.26×10^4

It is easy to show that Kramers' formula for the absorption coefficient is equivalent to taking

$$f'_{nn'} = \frac{2^6}{3\sqrt{3}\pi} \frac{1}{\varpi_{n'}} \frac{1}{\left(\frac{1}{n'^2} - \frac{1}{n^2}\right)^3} \left| \frac{1}{n^3} \frac{1}{n'^3} \right|. \tag{1.31}$$

When n is imaginary,

$$\left| \frac{1}{(i\kappa)^3} \right| = \frac{1}{\kappa^3}, \tag{1.32}$$

and similarly for n'. Equation (1.31) is a useful approximation in the discrete-discrete case, as well as for transitions involving continuous states. So close is the agreement between Kramers' expression and the exact expression derived from wave-mechanics, that we find it convenient to write

$$f_{nn'} = g f'_{nn'}, \tag{1.33}$$

where g is a correction factor depending upon n and n'. g is usually close to unity. From (1.15) and (1.33),

$$g = \pi\sqrt{3}D\left|\frac{[(n-n')/(n+n')]^{2n+2n'}nn'\Delta(n,n')}{(n-n')}\right|. \qquad (1.34)$$

We may now distinguish three cases, g_{I}, g_{II} and g_{III}, which refer respectively to the discrete-discrete, bound-free and free-free transitions. In these cases we have, accurately,

$$g_{\mathrm{I}} = \pi\sqrt{3}\frac{[(n-n')/(n+n')]^{2n+2n'}nn'\Delta(n,n')}{(n-n')}, \qquad (1.35)$$

$$g_{\mathrm{II}} = \frac{\pi\sqrt{3}\kappa n' e^{-4\kappa\tan^{-1}(n'/\kappa)}}{(\kappa^2+n'^2)^{1/2}(1-e^{-2\pi\kappa})}\,|\,\Delta(i\kappa,n')\,| \qquad (1.36)$$

and

$$g_{\mathrm{III}} = \frac{\pi\sqrt{3}\kappa l e^{-2\pi\kappa}}{(l-\kappa)(1-e^{-2\pi\kappa})(1-e^{-2\pi l})}\,|\,\Delta(i\kappa,il)\,|. \qquad (1.37)$$

When n' is large, the exact equations become somewhat cumbersome to handle. We have, accordingly, applied contour integration and the method of steepest descents to obtain the values of $|\Delta|$ in certain limiting cases. The details of the derivations are given in the Appendix. The results follow, with the conditions under which the formulæ hold indicated in parentheses.

$$g_{\mathrm{I}} \simeq \left[1 - \frac{0{\cdot}1728\left(1+\frac{n'^2}{n^2}\right)}{(1-n'^2/n^2)^{2/3}(n')^{2/3}} - \frac{0{\cdot}0496\left(1+\frac{484}{15}\frac{n'^2}{n^2}+\frac{n'^4}{n^4}\right)}{(1-n'^2/n^2)^{4/3}n'^{4/3}} + \right], \quad \left(\frac{n'}{n} << 1\right). \qquad (1.38)$$

$$g_{\mathrm{II}} \simeq \left[1 - \frac{0{\cdot}1728\left(1-\frac{n'^2}{\kappa^2}\right)}{(1+n'^2/\kappa^2)^{2/3}(n')^{2/3}} - \frac{0{\cdot}0496\left(1-\frac{484}{15}\frac{n'^2}{\kappa^2}+\frac{n'^4}{\kappa^4}\right)}{(1+n'^2/\kappa^2)^{4/3}(n')^{4/3}} + \right], \quad \left(\frac{\kappa}{n'} >> 1\right). \qquad (1.39)$$

$$g_{\mathrm{II}} \simeq \left[1 + \frac{0{\cdot}1728\left(1-\frac{\kappa^2}{n'^2}\right)}{(1+\kappa^2/n'^2)^{2/3}\kappa^{2/3}} - \frac{0{\cdot}0496\left(1-\frac{484}{15}\frac{\kappa^2}{n'^2}+\frac{\kappa^4}{n'^4}\right)}{(1+\kappa^2/n'^2)^{4/3}\kappa^{4/3}} + \right], \quad \left(\frac{\kappa}{n'} << 1\right). \qquad (1.40)$$

$$g_{\mathrm{III}} \simeq \left[1 + \frac{0{\cdot}1728\left(1+\frac{\kappa^2}{l^2}\right)}{(1-\kappa^2/l^2)^{2/3}\kappa^{2/3}} - \frac{0{\cdot}0496\left(1+\frac{484}{15}\frac{\kappa^2}{l^2}+\frac{\kappa^4}{l^4}\right)}{(1-\kappa^2/l^2)^{4/3}\kappa^{4/3}} + \right], \quad \left(\frac{l}{\kappa} >> 1\right). \qquad (1.41)$$

The following alternative expressions for the g factors, which take into account only the first two terms of the approximation and are valid only over the ranges indicated above, are easily obtained.

$$g_{\mathrm{I}} \simeq 1 - 0{\cdot}1728\left(\frac{\nu}{RZ^2}\right)^{1/3}\left[\frac{2}{n'^2}\left(\frac{RZ^2}{\nu}\right) - 1\right]$$

$$= 1 - 0{\cdot}1728\left(\frac{\nu}{RZ^2}\right)^{1/3}\left[\frac{2}{n^2}\left(\frac{RZ^2}{\nu}\right) + 1\right]. \qquad (1.42)$$

The formulæ for g_{II} and g_{III} may be deduced in a similar manner, or they may be obtained directly from (1.42) by allowing $n \to i\kappa$ and $n' \to il$, as before.

$$g_{II} \simeq 1 - 0{\cdot}1728\left(\frac{\nu}{RZ^2}\right)^{1/3}\left[\frac{2}{n'^2}\left(\frac{RZ^2}{\nu}\right) - 1\right]$$
$$= 1 + 0{\cdot}1728\left(\frac{\nu}{RZ^2}\right)^{1/3}\left[\frac{2}{\kappa^2}\left(\frac{RZ^2}{\nu}\right) - 1\right] \qquad (1.43)$$

and

$$g_{III} \simeq 1 + 0{\cdot}1728\left(\frac{\nu}{RZ^2}\right)^{1/3}\left[\frac{2}{l^2}\left(\frac{RZ^2}{\nu}\right) + 1\right]$$
$$= 1 + 0{\cdot}1728\left(\frac{\nu}{RZ^2}\right)^{1/3}\left[\frac{2}{\kappa^2}\left(\frac{RZ^2}{\nu}\right) - 1\right]. \qquad (1.44)$$

When $n' \to \infty$ in (1.40), which corresponds to the short-wave limit of the cont.-cont. spectrum, we have

$$g_{II}(i\kappa, \infty) = 1 + \frac{0{\cdot}1728}{\kappa^{2/3}} - \frac{0{\cdot}0496}{\kappa^{4/3}}. \qquad (1.45)$$

A comparison between the values of $g_{II}(i\kappa, \infty)$, as computed from (1.45), and those given by Gaunt in (6.45) is shown in Table III, where it is to

TABLE III

Values of $g_{II}(i\kappa, \infty)$

κ	Gaunt (6.45)	(1.17)
$\frac{1}{2}$	1·1	$1{\cdot}1_1$
1	1·2	$1{\cdot}1_2$
2	1·1	1·09

be noted that the last figures in Gaunt's values are stated to be uncertain. When $\kappa \to \infty$, we have from (1.39)

$$g_{II}(i\,\infty, n') = 1 - \frac{0{\cdot}1728}{(n')^{2/3}} - \frac{0{\cdot}0496}{(n')^{4/3}}. \qquad (1.46)$$

Values of $g_{II}(i\,\infty, n')$ computed from (1.46) are shown in Table IV, together with the values computed by Gaunt (6.50), Maue (Table 2), and the exact values obtained with the aid of equations (1.23) to (1.28). The exact values of $g_{II}(i\kappa, n')$, obtained from the latter equations, are plotted in fig. 1. Of these the curves for $n' = 1, 2, 3$ were given by Maue in his fig. 6 and were also tabulated by Strömgren.* The lines at the right of the diagram indicate the respective values of $g_{II}(i\,\infty, n')$.

* B. Strömgren, *Zs. für Astr.*, **4**, 118, 1932.

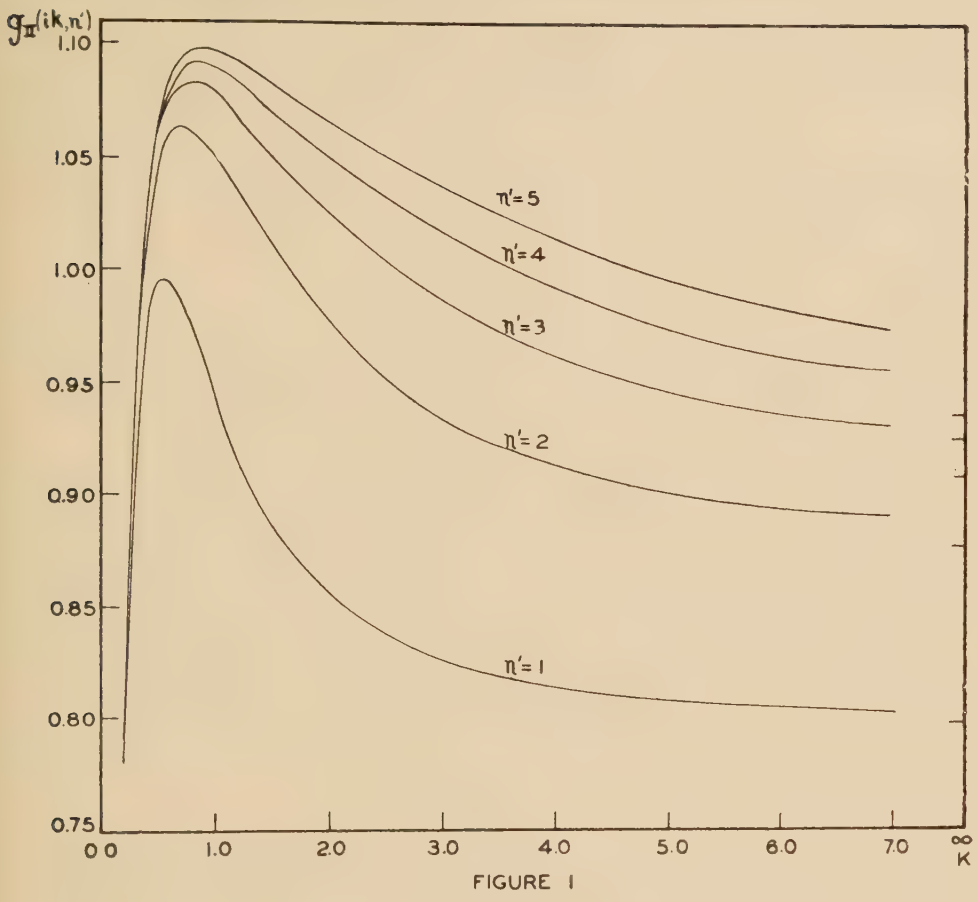

FIGURE 1

Values of g_{II} ($i\kappa$, n') computed from (1.36) *and* (1.23) *to* (1.27). *The lines at the right show the respective values of g_{II} ($i\infty$, n').*

TABLE IV

Values of $g_{II}(i\infty, n')$

n'	Exact	(1.18)	Maue	Gaunt
1	0·7973	0·7776	0·797	0·8
2	0·8762	0·8714	0·876	0·9
3	0·9075	0·9055	0·91	0·9
4	0·9247	0·9236	$0·92_5$	1·0
5	0·9358	0·9351	$0·93_5$	
6		0·9431	$0·94_5$	
7	0·9488	0·9491		

Further checks on our formulæ may be made in the limiting cases of $\kappa \to 0$ and when $\kappa \simeq l$. When $\kappa << 1$, $|\Delta(i\kappa, n')| \simeq |\Delta(i\kappa, il)| \simeq 8\kappa$,

and we have

$$g_{III} \simeq 4\sqrt{3}\kappa(1 - e^{-2\pi l})^{-1} \tag{1.47}$$

and

$$g_{II} \simeq 4\sqrt{3}\kappa, \tag{1.48}$$

which agree with Gaunt's equations (5.36) and (5.37) and with Maue's equations (47) and (50). When $\kappa \simeq l$, we have from A (26),

$$g_{III} \simeq \frac{\sqrt{3}}{\pi} ln\left(\frac{2\kappa}{e}\right) = \frac{\sqrt{3}}{\pi} ln\left(\frac{4\nu_\kappa}{\nu}\right), \tag{1.49}$$

in agreement with Gaunt (6.44) and Maue (61).

When $n - n'$ is small, the value of g_I may be computed most simply by equations A (27) and A (28). The approximation A (29) is also useful when n' is large.

At this point we should mention that the factor, $3\sqrt{3}$, by which Sugiura * multiplies g_{III}, is apparently erroneous. His derivation is supposed to take account of retardation, but elementary considerations show that this effect should lead to a correction term of the order v^2/c^2. Sugiura's work contains several errors, one of which is in the normalizing factors.

The quantities $A_{\kappa n'}$ for the bound-free transitions are related to the target areas for capture $\sigma_{\kappa n'}$ and to Milne's † capture probability coefficients $F(v)$. If $f(v)$ is the velocity-distribution function, the number of captures per cm.^{3} per second per velocity range dv is

$$N_i N_e f(v) v \sigma_{\kappa n'} dv, \tag{1.50}$$

where N_i is the number of ionized atoms per cm.^{3}.

In the present notation, the number of transitions from continuous states in the range $d\kappa$ corresponding to the velocity range dv is

$$N_\kappa A_{\kappa n'} d\kappa = N_i f(v) A_{\kappa n'} dv d\kappa, \tag{1.51}$$

whence

$$A_{\kappa n'} = N_e v \sigma_{\kappa n'} = 2\pi N_e v F(v) \tag{1.52}$$

and

$$\sigma_{\kappa n'} = \frac{2^4}{3\sqrt{3}} \frac{\epsilon^2 h}{m^2 c^3} \frac{1}{\left(\frac{1}{n'^2} + \frac{1}{\kappa^2}\right)} \frac{\kappa^2}{n'^3} g_{II}, \tag{1.53}$$

by (1.7), (1.31), (1.33) and the definition of κ:

$$\tfrac{1}{2} m v^2 = \frac{hRZ^2}{\kappa^2} = h\nu, \tag{1.54}$$

$$mvdv = -\frac{2hRZ^2}{\kappa^3} d\kappa = h d\nu. \tag{1.55}$$

In these equations, $h\nu$ obviously refers to the energy of the electrons and not to the frequency of the emitted radiation.

* Sugiura, *Sci. Pap. Inst. Phys. and Chem. Research (Tokyo)*, **17**, 89, 1931.

† Milne, *Phil. Mag.*, **47**, 209, 1924.

2. *The Atomic Coefficients of General Absorption*

We turn now to discussion of the absorption coefficients themselves. The continuous absorption coefficient per atom in state n' may be defined as follows :— *

$$a_{\kappa n'}(\nu) = \frac{\pi\epsilon^2}{mc}\frac{df_{\kappa n'}}{d\nu}, \tag{2.1}$$

where

$$df_{\kappa n'} = f_{\kappa n'}d\kappa. \tag{2.2}$$

Employing (1.31), (1.33), (1.54) and (1.55) we obtain

$$a_{n'}(\nu) = \frac{\pi\epsilon^2}{mc}\frac{1}{2RZ^2}\frac{2^5}{3\sqrt{3}\pi}\frac{1}{n'^5}\frac{1}{\left(\frac{1}{n'^2}+\frac{1}{\kappa^2}\right)^3}g_{II} = \frac{2^4\pi^2}{3\sqrt{3}}\frac{\epsilon^6Z^2}{ch^4\nu^3}\frac{1}{n'^2}\frac{2hRZ^2}{n'^3}g_{II}, \tag{2.3}$$

since

$$R = \frac{2\pi^2\epsilon^4 m}{h^3}. \tag{2.4}$$

If for the moment we omit the factor $(1 - e^{-h\nu/kT})$, which allows for the effect of negative absorptions, the total contribution at frequency ν to the optical depth $d\tau_\nu$ by atoms in the layer of thickness dH is

$$d\tau_\nu = \sum_{n_0'}^{\infty} N_{n'}a_{n'}(\nu)dH = N_0dH\frac{\varpi_\epsilon\varpi'}{b_0(T)}e^{-\chi_0/kT}\frac{2^4\pi^2}{3\sqrt{3}}\frac{\epsilon^6Z^2}{ch^4\nu^3}\cdot 2hRZ^2\times$$
$$\times\sum_{n_0'}^{\infty}\frac{e^{hRZ^2/n'^2kT}}{n'^3}g_{II}, \tag{2.5}$$

where N_0 is the total number of neutral atoms, irrespective of quantum state. $b_0(T)$ is the partition function of the neutral atom and χ_0 is the ionization potential,

$$b_0(T) = \sum_1^{\infty}\varpi_{n'}e^{-E_{n'}/kT}. \tag{2.6}$$

For hydrogen, ionized helium, etc., $\varpi_{n'}$ is given by (1.4), and $E_{n'}$, the excitation potential, is

$$E_{n'} = hRZ^2\left(1 - \frac{1}{n'^2}\right). \tag{2.7}$$

For complex atoms it is unlikely that equation (2.3) holds exactly for the lower atomic states. The upper atomic states of any atom are, however, essentially hydrogenic. If ϖ' is the weight of the ion, the total weight of all sub-levels produced by coupling an electron in all possible orbits of given total quantum number n' is

$$\varpi_{n'} = \varpi_e\varpi'n'^2. \tag{2.8}$$

If neutral atomic terms built on ionic states other than the lowest come into account, ϖ' in equation (2.8) should be replaced by $b'(T)$, the

* *Cf.* Menzel, *Publ. Lick Obs.*, **17**, 222, 1930.

partition function of the ion. The excitation potentials of the higher states will be

$$E_{n'} = \chi_0 - \frac{hRZ^2}{n'^2}. \tag{2.9}$$

This assumes zero quantum defect. n_0', the lower limit of the summation, is defined so that

$$\nu \geqslant \frac{RZ^2}{n_0'^2}. \tag{2.10}$$

Thus, in the case of hydrogen, if we are dealing with ν to the violet of $R/4$, the Balmer limit, $n_0' = 2$; if the ν under consideration lies between $R/4$ and $R/9$, $n_0' = 3$, etc. This equation may be further simplified if N_0 is eliminated by means of the equation of dissociative equilibrium,

$$\frac{N_i}{N_0} P_e = \frac{(2\pi m)^{3/2}(kT)^{5/2}}{h^3} \frac{\varpi_\epsilon b_1(T)}{b_0(T)} e^{-\chi_0/kT}. \tag{2.11}$$

Or

$$d\tau_\nu = N_i dH \frac{2^4\pi^2}{3\sqrt{3}} \frac{\epsilon^6 Z^2}{ch\nu^3} \frac{P_\epsilon}{(2\pi mkT)^{3/2}} \frac{2hRZ^2}{kT} \sum_{n_0'}^{\infty} \frac{e^{hRZ^2/n'^2kT}}{n'^3} g_{II}$$

$$= N_i dH C_0 \frac{Z^2 P_\epsilon T^{-3/2}}{\nu^3} \left[\frac{2hRZ^2}{\kappa T} \sum_{n_0'}^{\infty} \frac{e^{hRZ^2/n'^2kT}}{n'^3} g_{II} \right], \tag{2.12}$$

where

$$C_0 = \frac{2^4\pi^2}{3\sqrt{3}} \frac{\epsilon^6}{ch} \frac{1}{(2\pi mk)^{3/2}} = 2{\cdot}67 \times 10^{24}. \tag{2.13}$$

In the case of complex atoms it would appear to be permissible to evaluate the sum by means of integration. The energy levels formed by coupling an electron of total quantum number n' will not be coincident. The series levels of various atoms in a composite stellar atmosphere will overlap. Integration is equivalent to assuming a continuous set of quantum defects between successive integral quantum numbers:

$$\frac{2hRZ^2}{kT} \sum_{n_0'}^{\infty} \frac{e^{hRZ^2/n'^2kT}}{n'^3} g_{II} \simeq \frac{2hRZ^2}{kT} \int_{\sqrt{\frac{RZ^2}{\nu}}}^{\infty} \frac{e^{hRZ^2/n'^2kT}}{n'^3} g_{II} dn', \tag{2.14}$$

$$d\tau_\nu = N_i dH C_0 \frac{Z^2 P_\epsilon T^{-3/2}}{\nu^3} \bar{g}_{II} e^{h\nu/kT}, \tag{2.15}$$

$$\bar{g}_{II} \simeq 1 - 0{\cdot}1728 \left(\frac{\nu}{RZ^2}\right)^{1/3} \left(1 - \frac{2kT}{h\nu}\right) - e^{-h\nu/kT} \left[1 + 0{\cdot}1728 \left(\frac{\nu}{RZ^2}\right)^{1/3} \left(1 + \frac{2kT}{h\nu}\right)\right], \tag{2.16}$$

when use is made of (1.43). For the free-free transitions, the absorption coefficient per ionized atom for electronic transitions from one interval $d\nu'$ of electronic energy to another interval $d\nu$ becomes

$$\alpha_{\kappa,\, l}(\nu) = \frac{\pi\epsilon^2}{mc}\frac{d^2 f_{\kappa l}}{d\nu d\nu'}, \tag{2.17}$$

where

$$d^2 f = f_{\kappa l} d\kappa dl. \tag{2.18}$$

Employing (1.31)–(1.33), (1.54), (1.55) and

$$\tfrac{1}{2}mv^2 = \frac{hRZ^2}{l^2} = h\nu', \tag{2.19}$$

we obtain

$$\alpha_{\kappa l}(\nu) = \frac{\pi\epsilon^2}{mc}\,\frac{1}{4R^2Z^4}\,\frac{2^6}{3\sqrt{3}\pi}\,\frac{1}{\varpi_e}\,\frac{1}{\left(\frac{1}{\kappa^2}-\frac{1}{l^2}\right)^3} g_{\mathrm{III}}. \tag{2.20}$$

In this case the optical depth at ν is obtained by integrating over all values of l (*i.e.* over ν'), with g_{III} as given in equation (1.44).

$$d\tau_\nu = N_i dH \frac{2^5\pi^2\epsilon^6 Z^2}{3\sqrt{3}ch^3\nu^3}\int_0^\infty \frac{f(v)dv g_{\mathrm{III}}}{\varpi_e} d\nu' = N_i dH C_0 \frac{Z^2 P_e T^{-3/2}}{\nu^3}\int_0^\infty e^{-h\nu'/kT} g_{\mathrm{III}} d\left(\frac{h\nu'}{kT}\right) \tag{2.21}$$

in thermodynamic equilibrium. $f(v)$ is given by Maxwell's law :

$$f(v)dv = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}mv^2/kT} v^2 dv. \tag{2.22}$$

If we express the electron pressure in dynes per cm.^{2},

$$P_e = N_e kT. \tag{2.23}$$

Introducing equation (1.44), we find the following expression :—

$$d\tau_\nu = N_i dH C_0 \frac{Z^2 P_e T^{-3/2}}{\nu^3}\bar{g}_{\mathrm{III}}, \tag{2.24}$$

where

$$\bar{g}_{\mathrm{III}} = 1 + 0{\cdot}1728\left(\frac{\nu}{RZ^2}\right)^{1/3}\left[1 + \frac{2kT}{h\nu}\right]. \tag{2.25}$$

The effective absorption coefficient $\alpha'(\nu)$ *per ionized atom* is obtained by summing the contributions of the bound-free and free-free transitions and multiplying by $(1 - e^{-h\nu/kT})$ to allow for the negative absorptions. Thus equations (2.15) and (2.24) give

$$\alpha'(\nu) = \frac{C_0 Z^2 P_e}{\nu^3 T^{3/2}}(e^{h\nu/kT} - 1)g', \tag{2.26}$$

where

$$g' \simeq 1 - 0{\cdot}1728\left(\frac{\nu}{RZ^2}\right)^{1/3}\left(1 - \frac{2kT}{h\nu}\right). \tag{2.27}$$

For hydrogen-like atoms we may write

$$\alpha'(\nu) = \frac{C_0 Z^2 P_e}{\nu^3 T^{3/2}}\left[\frac{2hRZ^2}{kT}\sum_{n_0'}^{\infty}\frac{e^{hRZ^2/n'^2kT}}{n'^3} g_{\mathrm{II}} + \bar{g}_{\mathrm{III}}\right](1 - e^{-h\nu/kT}) \tag{2.28}$$

from (2.12) and (2.24). Equations (2.26) and (2.28) should be employed in place of those given by Pannekoek,* whose formulæ are in error through omission of a weight factor n'^2 in (2.14) *et seq.* He also sets $g = 1$ throughout.

It is interesting to apply equation (2.26) to the calculation of opacity in the solar atmosphere. We shall assume that hydrogen is the predominant constituent, but that most of the opacity comes from metallic atoms, for which $Z = 1$. The validity of this last assumption will be discussed further in Section 3. Under these physical conditions the total optical depth down to a layer where the electron pressure is P_0 is given by †

$$\tau_1 = \frac{1}{\bar{m}g}\frac{C_0 P_0^2}{\nu^3 T^{3/2}}(e^{h\nu/kT} - 1)g'\left[1 - \frac{K}{P_0} + \frac{K^2}{P_0^2}\log\left(1 + \frac{P_0}{K}\right)\right], \qquad (2.29)$$

where $\bar{m}$ is the mean atomic mass, here taken to be that of hydrogen, and g is the surface gravity, $2{\cdot}74 \times 10^4$ cm./sec.$^{-2}$. K is the dissociation constant of the opacity-producing ion; it is equivalent to the right-hand side of equation (2.11). For *Mg* and *Fe*, the most abundant of the metals, $K > P$, for solar conditions, and the bracketed factor is practically $\frac{1}{2}$. Milne ‡ has shown that the effective photosphere lies approximately at $\tau_1 = \frac{1}{3}$. Taking $\nu = 6 \times 10^{14}$, which corresponds approximately to λ 5000 A., and $T = 6000°$, we find $P_0 = 32$ dynes or $3{\cdot}2 \times 10^{-5}$ atmospheres. From an analysis of the relative intensities of lines from ionized and neutral lines of the same element, Russell § derives the value $P_0 = 46 \pm 9$ dynes. The close agreement between observed and calculated values is encouraging indeed. The present result is in good agreement with an earlier result obtained by one of us,‖ wherein a value of $P_0 = 40$ dynes was deduced. The same general wave-mechanical formulæ, however, formed the basis of both computations.

3. *Mean Values of the Absorption Coefficient*

Equation (2.26) may also be used to calculate a_R', the reciprocal or Rosseland mean absorption coefficient, defined as follows :—

$$a_R' = \frac{\int_0^\infty \frac{\partial}{\partial T}\frac{\nu^3}{e^{h\nu/kT} - 1}d\nu}{\int_0^\infty \frac{1}{a'(\nu)}\frac{\partial}{\partial T}\frac{\nu^3}{(e^{h\nu/kT} - 1)}d\nu} = \frac{C_0 h^3 Z^2 P_e}{k^3 T^{9/2}}\frac{24\theta(4)}{7!\,\frac{1}{2}[\theta(6) - \theta(7)]}g_R', \qquad (3.1)$$

where

$$g_R' = 1 - \frac{0{\cdot}1728\left(\frac{kT}{hRZ^2}\right)^{1/3}\{\Gamma(25/3)[\theta(19/3) - \theta(22/3)] - 2\Gamma(22/3)[\theta(16/3) - \theta(19/13)]\}}{7!\,[\theta(6) - \theta(7)]}$$

$$= 1 - 0{\cdot}1098\left(\frac{kT}{hRZ^2}\right)^{1/3}, \qquad (3.2)$$

* Pannekoek, *M.N.*, **91**, 162, 1931.

† Russell, *Ap. J.*, **78**, 239, 1933; *Contr. Mt. Wilson Obs.*, No. 477.

‡ Milne, *Phil. Trans.*, A, **228**, 421, 1929.

§ Russell, *op. cit.*; *cf.* also *Ap. J.*, **70**, 11, 1929; *Contr. Mt. Wilson Obs.*, No. 383.

‖ Menzel, *Publ. Lick Obs.*, **17**, 245, 1930.

and

$$\left.\begin{aligned} &\theta(a) = 1 + \frac{1}{2^a} + \frac{1}{3^a} + \ldots, \\ &\theta(4) = \frac{\pi^4}{90}, \qquad \theta(6) = 1{\cdot}017343, \qquad \theta(7) = 1{\cdot}008349, \\ &\theta(\tfrac{16}{3}) = 1{\cdot}028597, \qquad \theta(\tfrac{19}{3}) = 1{\cdot}013564, \qquad \theta(\tfrac{22}{3}) = 1{\cdot}006567. \end{aligned}\right\} \quad (3{\cdot}3)$$

Thus, we have numerically

$$a_R' = C_R \frac{Z^2 P_e}{T^{9/2}} g_R' \quad (3{\cdot}4)$$

$$g_R' = 1 - 0{\cdot}1098\left(\frac{T}{1{\cdot}57 \times 10^5 Z^2}\right)^{1/3} \quad (3{\cdot}5)$$

$$C_R = \frac{16\pi^2\theta(4)}{315\sqrt{3}[\theta(6) - \theta(7)]} \frac{\epsilon^6 h^2}{c\kappa^3} \frac{1}{(2\pi m k)^{3/2}} = 3{\cdot}32 \times 10^{-7}. \quad (3{\cdot}6)$$

The final result should be valid as long as the second term in g_R' is small.

With the exception of the small correction factor g_R', equation (3.4) is identical with that given by Rosseland,* derived by quite different logic and stated by him to be an upper limit to the value of the opacity coefficient. Chandrasekhar † criticizes this value on several counts, among which are the following :—

(1) Rosseland's method amounts to an extrapolation of the Kramers-Gaunt formula for free-free transitions to the case of the bound-free transitions.

(2) The assumption of a continuous series of negative energy states is fictitious and, since the Rosseland mean introduces a weighting factor $\nu^4 e^{-h\nu/kT}$, the resultant value of the absorption coefficient will be too large.

(3) The extension of the Maxwellian velocity distribution to states of negative energy leads to a gross exaggeration of the absorption arising from bound states of large negative energy.

(4) The total absorption at frequency ν yields an infinite mean ‡ absorption coefficient.

Chandrasekhar's criticisms do not appear to be well founded. First, the extrapolation of the free-free to the bound-free case is seen to be justified. Second, the convention of adopting a continuous series of negative states is equivalent to our substitution of an integration for the more accurate summation. Further, the weighting factor proves to be exceptionally low. From equations (2.26) and (3.1) we find that the value of $\bar{\nu}$, corresponding to the average frequency used in obtaining the Rosseland mean, is given by

$$\frac{\bar{\nu}^3}{(e^{h\bar{\nu}/kT} - 1)} = \left(\frac{kT}{h}\right)^3 \frac{7!\,\frac{1}{2}[\theta(6) - \theta(7)]}{24\theta(4)}, \quad (3{\cdot}7)$$

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* Rosseland, *Astrophysik*; *cf.* also *Handb. der Astrophysik*, Band III, Part 1, pp. 469 *et seq.*

† Chandrasekhar, *Proc. Roy. Soc.*, A, **133**, 241, 1931 ; *ibid.*, **135**, 472, 1932.

‡ The "straight," not the Rosseland mean.

whence

$$\bar{\nu} \simeq 1{\cdot}36\frac{kT}{h}, \quad \text{or} \quad \lambda \simeq 1{\cdot}05T. \tag{3.8}$$

This value of $\bar{\nu}$ is well within the limit where (2.26) would be expected to fail seriously, and the formula should be quite accurate. Third, the extension of the Maxwellian distribution to negative energy states is quite proper, since it is nothing more than an expression of the Boltzmann law. Finally, the infinite value of the straight mean absorption coefficient arises from the extension of the integration to values of $h\nu \geqslant \chi_0$, where χ_0 is the ionization potential of the atom in question. An approximate value of the absorption coefficient for frequencies in this range is obtained by setting $h\nu = \chi_0$ in the exponential term of (2.26). Our failure to do this in the case of the Rosseland mean, however, introduces no appreciable error in the result, since the exponential then falls in the denominator.

The chief objection that may be urged against the validity of the formulæ (2.26) and (2.28) is the use we have made of the Boltzmann and dissociation formulæ, since these are strictly applicable only in the case of thermodynamic equilibrium. The absorption coefficient is usually employed to calculate the *flux* of radiation, which in turn violates the condition of thermodynamic equilibrium, for which state the flux is zero. Nevertheless, the first appreciable deviation from thermodynamic equilibrium occurs in the lowest atomic states, which do not enter into the formulæ (2.26) and (2.28). To the first approximation the partition of atoms among the higher atomic states may be represented as a Boltzmann distribution characterized by a parameter T_e, the electron temperature, instead of T_0, the temperature of the radiation.

The absorption coefficient depends upon the value of Z. In stellar atmospheres, where most of the opacity comes from the ionization of neutral atoms, $Z = 1$. Russell * sets $Z = (\chi_0/R)^{\frac{1}{2}}$, where χ_0 is the ionization potential of the opacity-producing atom. The opacity we are considering, however, comes chiefly from atomic states of high total and high azimuthal quantum number, for which the energy levels are essentially hydrogenic.

It is interesting to apply our results to an assembly consisting of ionized complex atoms. Consider an atom of atomic number Z and therefore capable of existing in $Z+1$ different stages of ionization. Denote the number of electrons that have been removed from a given atom by q and the percentage of such atoms by x_q. Then, at a given temperature and density, the various equilibria become

$$\frac{x_1}{x_0} = \frac{K_1}{P}, \qquad \frac{x_2}{x_1} = \frac{K_2}{P}, \qquad \cdots \frac{x_q}{x_{q-1}} = \frac{K_2}{P}, \tag{3.9}$$

where P is the electron-pressure and

$$K_q = \frac{(2\pi m)^{3/2}(kT)^{5/2}}{h^3} \frac{\varpi_e b_q(T)}{b_{q-1}(T)} e^{-\chi_{q-1}/kT}. \tag{3.10}$$

* Russell, *Ap. J.*, **78**, 239, 1927.

The notation is analogous to that used in equation (2.11). Thus

$$x_q = x_0 \frac{K_1 K_2 \ldots K_q}{P^q} \tag{3.11}$$

and

$$\sum_{q=0}^{Z} x_q = 1. \tag{3.12}$$

We easily find that

$$x_0 = \frac{1}{\Phi}, \tag{3.13}$$

where

$$\Phi = 1 + \frac{K_1}{P} + \frac{K_1 K_2}{P^2} + \ldots \frac{K_1 K_2 \ldots K_Z}{P^Z}. \tag{3.14}$$

Also

$$P = N(x_1 + 2x_2 + \ldots Zx_Z)kT. \tag{3.15}$$

N is the total number of atoms per cm.^{3} in all stages of ionization. The optical depth arising from a gas of this type is, by equation (2.5) *et seq.*,

$$d\tau = NdH \sum_{0}^{Z-1} x_q \varpi_e \frac{b_{q+1}(T)}{b_q(T)} e^{-\chi_q/kT} \frac{2^4\pi^2}{3\sqrt{3}} \frac{\epsilon^6(q+1)^2}{ch^4\nu^3} kTe^{h\nu/kT}, \tag{3.16}$$

where we have set $g' = 1$. This equation includes the free-free transitions but not the stimulated emissions. For accurate solution of the problem it is necessary to know all the ionization potentials, χ_q. Approximate values have been assigned in certain cases, but the mathematical relationships between successive ionization potentials is quite complex. We may, however, consider an idealized atom, where

$$\chi_q = hR(q+1)^2. \tag{3.17}$$

This gives the correct value for

$$\chi_{Z-1} = hRZ^2, \tag{3.18}$$

and also the correct order of magnitude for χ_0. Writing

$$K_q = ae^{-bq^2}, \tag{3.19}$$

and setting the factor $\omega_e b_q(T)/b_{q-1}(T)$ equal to unity, we find

$$\Phi = 1 + \frac{a}{P}e^{-b} + \frac{a^2}{P^2}e^{-b(1+4)} + \ldots + \frac{a^q}{P^q}e^{-b(1+4+\cdots+q^2)} + \ldots = \sum_{0}^{Z} e^{q\ln\frac{a}{P} - \frac{b}{3}\left(q^3+\frac{3}{2}q^2+\frac{q}{2}\right)} \tag{3.20}$$

$$\simeq \int_0^\infty e^{q\ln\frac{a}{P} - \frac{b}{3}\left(q^3+\frac{3}{2}q^2+\frac{q}{2}\right)} dq, \tag{3.21}$$

as long as the atoms of the assembly are not completely "stripped." To evaluate this integral we first determine q_0, the value of q for which the exponent has a maximum:

$$q_0 = -\frac{1}{2} + \left(\frac{1}{12} + \frac{\ln a/P}{b}\right)^{\frac{1}{2}}. \tag{3.22}$$

By means of the substitution

$$y \equiv (q-q_0)\left(\frac{b^2}{12}+b\ln\frac{a}{P}\right)^{\frac{1}{4}},$$

the integral of (3.21) may be expressed in the form

$$\Phi = \frac{\left(\frac{a}{P}\right)^{q_0} e^{-\frac{b}{3}\left(-\frac{1}{6}+\frac{\ln\frac{a}{P}}{b}\right)\left(\frac{1}{12}+\frac{\ln\frac{a}{P}}{b}\right)^{\frac{1}{2}}}}{\left(\frac{b^2}{12}+b\ln\frac{a}{P}\right)^{\frac{1}{4}}} \int_{-y_0}^{\infty} e^{-y^2-\frac{y^3}{\alpha}} dy, \tag{3.23}$$

where

$$\alpha \equiv 3b^{\frac{1}{2}}\left(\frac{1}{12}+\frac{\ln\frac{a}{P}}{b}\right)^{\frac{3}{4}} \tag{3.24}$$

and

$$y_0 = q_0[b(q_0+\tfrac{1}{2})]^{\frac{1}{2}}. \tag{3.25}$$

If y_0 and α are large,

$$\int_{-y_0}^{\infty} e^{-y^2-y^3/\alpha} dy \simeq \int_{-\infty}^{\infty} e^{-y^2}\left(1-\frac{y^3}{\alpha}+\frac{y^6}{2\alpha^2}-\frac{y^9}{6\alpha^3}+\frac{y^{12}}{24\alpha^4}\cdots\right)dy$$

$$= \sqrt{\pi}\left[1+\frac{15}{16\alpha^2}+\frac{15}{16}\frac{(7\cdot 9\cdot 11)}{8\cdot 24\alpha^4}+\cdots\right], \tag{3.26}$$

whence

$$\Phi = \frac{\left(\frac{a}{P}\right)^{q_0} e^{-\frac{b}{3}\left(-\frac{1}{6}+\frac{\ln\frac{a}{P}}{b}\right)\left(\frac{1}{12}+\frac{\ln\frac{a}{P}}{b}\right)^{\frac{1}{2}}} \sqrt{\pi}}{\left(\frac{b^2}{12}+b\ln\frac{a}{P}\right)^{\frac{1}{4}}}\left[1+\frac{15}{16\alpha^2}+\cdots\right]. \tag{3.27}$$

Similarly

$$P = \frac{NkT}{\Phi}\sum_0^Z q\left(\frac{a}{P}\right)^q e^{-\frac{b}{3}\left(q^3+\frac{3}{2}q^2+\frac{q}{2}\right)} = \frac{NkT}{\Phi}\frac{\partial\Phi}{\partial\ln\left(\frac{a}{P}\right)} = NkT\left(q_0-\frac{1}{\frac{b}{3}+4\ln\frac{a}{P}}\right). \tag{3.28}$$

$$\alpha_\nu = \frac{24\pi^2}{3\sqrt{3}}\frac{\epsilon^6}{ch^4\nu^3}kTe^{h\nu/kT}\frac{1}{\Phi}\sum^{Z-1}\left(\frac{a}{P}\right)^q (q+1)^2 e^{-\frac{b}{3}\left(q^3+\frac{3}{2}q^2+\frac{q}{2}\right)-b(q+1)^2}. \tag{3.29}$$

Using the same method of asymptotic integration as in (3.26), we obtain

$$\sum_0^{Z-1}\left(\frac{a}{P}\right)^q (q+1)^2 e^{-\frac{b}{3}\left(q^3+\frac{3}{2}q^2+\frac{q}{2}\right)-b(q+1)^2} \cong \left(1+\frac{2\partial}{\partial\ln\left(\frac{a}{P}\right)}+\frac{\partial^2}{\partial\ln\left(\frac{a}{P}\right)^2}\right)\times$$

$$\times\int_0^{\infty} e^{q\ln(a/P)-\frac{b}{3}\left(q^3+\frac{3}{2}q^2+\frac{q}{2}\right)-b(q+1)^2} dq \cong \frac{\Phi}{\left(\frac{a}{P}\right)}\left[\frac{\ln\left(\frac{a}{P}\right)}{b}-\sqrt{\frac{1}{12}+\frac{\ln(a/P)}{b}}+\frac{1}{3}\right]. \tag{3.30}$$

If we now multiply (3.29) by $(1-e^{-h\nu/kT})$ to take account of the stimulated emission, and by (3.7) use $0{\cdot}873\left(\frac{kT}{h}\right)^3$ for $\frac{\bar{\nu}^3}{(e^{h\bar{\nu}/kT}-1)}$, we get

$$\bar{a}_R=\frac{2^4\pi^2}{3\sqrt{3}}\,\frac{\epsilon^6kT\left[\frac{ln(a/P)}{b}-\sqrt{\frac{1}{12}+\frac{lna/P}{b}}+\frac{1}{3}\right]}{ch^4 0{\cdot}873\left(\frac{kT}{h}\right)^3\left(\frac{a}{P}\right)}. \qquad (3.31)$$

We shall now apply (3.28) and (3.31) to the centre of Capella. Using Eddington's data,* $T=7{\cdot}2\times10^6$, $\rho=0{\cdot}0547$ and $N=5{\cdot}88\times10^{20}$ for the number of (iron) atoms per cm.^{3}, we get from (3.28) $P=1{\cdot}11\times10^{13}$ dynes/cm.^{2}. q_0, the most abundant stage of ionization, is 19. From (3.31) we obtain $a_R=1{\cdot}86\times10^{-22}$ for the mean absorption coefficient per atom. The absorption coefficient per gram $\kappa=a_RN/\rho=2{\cdot}00$, as compared with Eddington's value of 4·95 (*loc. cit.*, p. 229). The largest term in the sum of (3.29) is the 19th ($q=18$), so that the effective value of Z is about 19. Eddington uses a value of 26 for Z, which introduces a factor of two in κ, since a_ν is proportional to Z^2. Agreement with the astronomical value of 50 is no longer expected. The method of extending these formulæ to the cases discussed by B. Strömgren, who assumes an excess of H, is obvious.

The effect upon the mean absorption coefficient that arises from abrupt changes in opacity at each series limit has been widely discussed. The absorption beyond the series limits runs continuously into the overlapping lines to the redward of the limit. Furthermore, at the extremely high temperatures and densities of stellar interiors it appears probable that even the lowest atomic levels will be appreciably broadened. Probably there is no line absorption at all. All of the series members overlap and form a continuum, the effect of which should be added to that of the ordinary continuous absorption. Though the magnitude of the correction cannot be estimated with precision, elementary considerations based on the value of the ratio $\int df : \sum f$ for the predominant series indicate that the factor may be as much as 2 or 3.

Here there is no difficulty with the so-called "guillotine effect," since the lower limit, $n=1$, is specified and the pseudo-classical quantum range $0 \leqslant n < 1/2$ does not come into account.

4. *Absorption Coefficient for Neutral Hydrogen Atoms in Stellar Atmospheres*

Pannekoek † has suggested that part, if not all, of the opacity of stellar atmospheres, especially for the later-type stars, may come from the interaction of neutral hydrogen atoms with free electrons. The electrons are supposed to come from the more easily ionized metals, and tend to interact

* Eddington, *Internal Constitution of the Stars*, p. 229.

† Pannekoek, *op. cit.*

with the neutral hydrogen rather than with their own positive ions because of the relatively high abundance of the former. Russell,* who found Pannekoek's formula, the analogy of our equation (2.26), inadequate to account for the observed opacity, has also discussed the possibility of a contribution by neutral atoms to the atmospheric opacity. We have shown that our formula removes the old discrepancy between theory and observation. Nevertheless, it would appear worth while to investigate the absorption effects of neutral atoms.

The problem of the continuous emission during the free-free transitions of electrons colliding with neutral atoms has been treated by L. Nedelsky.† He assumes that the potential field V of the atoms is given by

$$\begin{aligned} V &= Ze^2/a - Ze^2/r \qquad \text{for } r \leqslant a \\ &= 0 \qquad\qquad\qquad\quad \text{for } r > a, \end{aligned} \tag{4.1}$$

which is equivalent to the field due to a point charge Ze surrounded by a spherical shell of radius a carrying a charge $-Ze$. His formulæ can be applied directly to our case by making $Z=1$ and taking the radius of the first Bohr orbit for a. Now in the spectroscopically accessible range of frequencies the effective range of electron velocities for temperatures between 3000° A. and 8000° A. is about from $0{\cdot}7 \times 10^8$ to $1{\cdot}5 \times 10^8$ cm./sec. In this velocity range Nedelsky's approximation for slow electrons is inapplicable, since his quantity y, which is supposed to be much less than one, varies here from about 0·3 to 0·7. In using his exact formula (24) it was found, however, that the first two terms in the summation were sufficient :

$$\mathcal{J}_\nu = \frac{16e^2h^2n'^2}{3c^3\pi^3m^2}\frac{y'}{y}[\,|\,\delta_0'\,|^2\,|\,\delta_1\,|^2\,|\,I_{1,0}\,|^2 + |\,\delta_1'\,|^2\,|\delta_0|^2\,|I_{0,1}\,|^2\,], \tag{4.2}$$

where $\mathcal{J}_\nu$ denotes the radiation emitted by an electron during a single encounter with an atom. This is in accordance with his observation on p. 653 that the contribution from the terms with $1 > y$ is negligible. Further, it was found semi-empirically that the individual factors appearing in the brackets of (4.2) could be represented for the effective velocity range by linear or quadratic expressions in v^2 and v'^2, v and v' being the initial and final velocities respectively. Thus the quantity in brackets was conveniently expressed by a polynomial of the sixth order in v'^2 with coefficients depending on v^2. Since $n'^2 \simeq 0{\cdot}506(1 + 0{\cdot}105v'^2 + 0{\cdot}0111v'^4)$, and $(y'/y) = (v'/v)$, we have

$$\mathcal{J}_\nu = \frac{4e^2h^2}{3c^3\pi m^2}(v'/v)P(v, v'^2), \tag{4.3}$$

P standing for $(\pi^2/4)n'^2$ times the brackets in (4.2).

We wish to compute the absorption coefficient per hydrogen atom in a gas containing N_e electrons per unit volume. We do this, in the usual way,‡ by first computing the emission. Since an electron of velocity v

* Russell, *Ap. J.*, **78**, 239, 1927.
† Nedelsky, *Phys. Rev.*, **42**, 641, 1932.
‡ Milne, *M.N.*, **85**, 750, 1925.

emits light of all frequencies up to $\nu = mv^2/2h$, we have for the intensity of radiation, E_ν,

$$E_\nu d\nu = 4\pi N_e \int_{v_0}^{\infty} (m/2\pi kT)^{3/2} e^{-mv^2/2kT} v^3 \mathcal{J}_\nu(v^2, v'^2) dv d\nu =$$

$$= 4\pi N_e (4e^2h^2/3c^3\pi m^2)(m/2\pi kT)^{3/2} \int_{v_0}^{\infty} e^{-mv^2/2kT} v^2 v' P(v^2, v'^2) dv d\nu, \qquad (4.4)$$

where $v_0^2 \equiv \dfrac{2h\nu}{m} = v^2 - v'^2$. In thermodynamic equilibrium,

$$E_\nu d\nu = 4\pi I_\nu \alpha_\nu d\nu = 4\pi (2h\nu^3/c^2) e^{-h\nu/kT} \alpha_\nu d\nu, \qquad P_e = N_e kT.$$

Hence

$$\alpha_\nu = \frac{2e^2 h P_e (m/2\pi kT)^{3/2}}{3c\pi m^2 \nu^3 kT} e^{h\nu/kT} \int_{v_0}^{\infty} e^{-mv^2/2kT} v^2 v' P(v^2, v'^2) dv d\nu.$$

Let

$$y \equiv v'^2/v_0^2, \qquad z \equiv (mv_0^2/2kT),$$

then

$$e^{h\nu/kT} \int_{v_0}^{\infty} e^{-mv^2/2kT} v^2 v' P(v^2, v'^2) dv = (v_0^4/2) \int_0^{\infty} e^{-zy} (y + y^2)^{\frac{1}{2}} P(v_0^2, v_0^2 y) dy \equiv$$

$$\equiv (v_0^4/2) A(z, v_0^2), \qquad (4.5)$$

where we have first transformed $P(v^2, v'^2)$ into $P(v_0^2, v'^2)$ through the relation $v^2 = v_0^2 + v'^2$, and then written $v_0^2 y$ for v'^2. Integrals of the form $\int_0^{\infty} e^{-zy} y^n (y + y^2)^{\frac{1}{2}} dy \equiv \psi_n(z)$ can be expressed in terms of Hankel functions of the first kind with imaginary arguments, for which extensive tables exist.* We have

$$\psi_0(x) = -(\pi/4x) e^{x/2} H_1^{(1)}\left(\frac{ix}{2}\right),$$

$$\psi_1(x) = -\frac{d\psi_0(x)}{dx} = \psi_0(x)[(2/x) - (\tfrac{1}{2})] + (i\pi/8x) e^{x/2} H_0^{(1)}(ix/2). \qquad (4.6)$$

The higher-order functions may be obtained from the identity

$$\psi_n(x) = (-)^n \frac{d^n \psi_0(x)}{dx^n},$$

and the relations

$$\frac{d^n \psi_0(x)}{dx^n} = [1 - (n+1)/x] \frac{d^{n-1} \psi_0(x)}{dx^{n-1}} + [(n - \tfrac{1}{2})/x] \frac{d^{n-2} \psi_0(x)}{dx^{n-2}}.$$

Thus we computed the ψ's for a set of values of z corresponding to the temperature and velocity ranges in Table V. As a check on the calculations, we developed asymptotic series for the ψ's for large values of z and found that the results agree. Further, the contribution from the high-order terms

* Jahnke and Emde, *Funktionen Tafeln*; G. N. Watson, *Bessel Functions*.

in P becomes increasingly small, which substantiates the validity of the original approximation for $\mathcal{J}_\nu$. Values of a_ν computed from

$$a_\nu = \frac{2e^2hP_e(m/2kT)^{1/2}z^2A(z, v_0^2)}{3c\pi^{5/2}m^3\nu^3} = aP_e \tag{4.7}$$

are given in Table V. This should be multiplied, also, by the factor $(1 - e^{-h\nu/kT})$ to allow for the negative absorptions.

TABLE V

Values of a_ν in units of $10^{-28}P_e$. [Correct to within 5 per cent.]

T \ λ	8900 A.	6810 A.	5390 A.	4360 A.	3030 A.
3000°	6·82	4·12	2·70	1·88	1·04
4000	5·81	3·41	2·20	1·50	0·81
6000	4·83	2·71	1·69	1·12	0·55
8000	4·33	2·36	1·44	0·93	0·47

Russell has derived the following formula for τ_2, the optical depth arising from the interaction of electrons with neutral atoms:—

$$\tau_2 = \frac{aP_0^2}{2\bar{m}g\epsilon}(1 - e^{-h\nu/kT})\left(\frac{4}{3}\frac{P_0}{K} + 1\right), \tag{4.8}$$

where the neutral atoms are assumed to be in great abundance. ϵ is the relative abundance, by volume, of the element giving rise to the electrons. From equation (2.29) we have

$$\frac{\tau_2}{\tau_1} \simeq \frac{a\nu^3T^{3/2}}{c_0\epsilon g'}e^{-h\nu/kT}\left(\frac{4}{3}\frac{P_0}{K} + 1\right). \tag{4.9}$$

For metallic atoms in the sun $\epsilon \simeq 10^{-3}$ and $P_0/K << 1$. Hence, for $\lambda = 5000°$A.,

$$\frac{\tau_2}{\tau_1} = 6{\cdot}5 \times 10^{-2}. \tag{4.10}$$

For a star where $T \sim 3000°$ a much greater percentage of atoms will be neutral. *Na* and *K* are the only atoms of any abundance that approach complete ionization, and for these atoms $\epsilon \sim 10^{-5}$. The change in the other factors, however, appears to offset the effect, for we find, with $P_0/K \sim 10$,

$$\frac{\tau_2}{\tau_1} \sim 0{\cdot}25. \tag{4.11}$$

The exact values are very sensitive to the factors assumed for the abundances. The marked spectral differences between dwarf and giant M stars may possibly be due to the opacity of neutral atoms operating in the former. If we may rely on the values of ϵ, then the effect of neutral hydrogen on the opacity is not important for stars of spectral types G5 and earlier.

We should, however, take such opacity into consideration in the later types, especially for members of the dwarf sequence.

5. *Discussion of Woolley's Work on Absorption Coefficients*

As this paper nears completion there comes to hand a discussion, by Woolley,* of the general problem of continuous absorption coefficients. He introduces an oscillator strength, f_c, for the continuum :

$$f_c = \int df = \int_{RZ^2/n^2}^{\infty} \frac{df}{d\nu} d\nu = \frac{8}{3\sqrt{3}\pi n} \bar{g} \tag{5.1}$$

by equation (2.3). A mean value of g is employed. Woolley substitutes f_c in place of part of the constant of $\frac{df}{d\nu}$.

The resulting absorption coefficient, for hydrogen, becomes

$$a_\nu = \frac{2\pi\epsilon^2}{mc\nu^3} \left(\frac{R}{n^2}\right)^2 f_c. \tag{5.2}$$

In applying this formula to complex spectra Woolley replaces $(R/n^2)^2$ by $\nu_0{}^2$, the particular series limit under consideration. From the equation,

$$\sum_n f_{nn'} + f_c = 1, \tag{5.3}$$

and from Korff's † experiments, which indicate a value of f very close to unity for the D lines, Woolley concludes that f_c must be very small. The Kramers formula applied to continuous absorption beyond the limit of the principal series of sodium will give, therefore, too large a result, unless, as Woolley suggests, a large g value is adopted. We note that equations (5.1) and (5.3) would rule out even this hope of escape. We cannot agree, however, with the sweeping generalization that Woolley draws from this single example : that Kramers' formula gives too high an opacity for optical spectra of all complex atoms.

The absorption beyond the principal series is of no particular significance in the astronomical opacity problem ; it comes into account only where rates of ionization are important. The transfer of radiation through the atmospheric layers of the star is mainly governed by continuous absorption by atoms in atomic states only two or three volts below the ionization level. These states are practically hydrogenic, and the quantum defects will be practically zero for the states of greatest weight (highest value of azimuthal quantum number). Korff's result cannot be used as an argument against the validity of the formulæ we give in the preceding sections.

Woolley attempts to evaluate f_c for the metals on the basis of a most surprising assumption. Milne ‡ has found that the spectral distribution of

* Woolley, *M.N.*, **95**, 101, 1934.
† Korff, *Ap. J.*, **76**, 124, 1932.
‡ Milne, *Phil. Trans.*, A, **223**, 213, 1922.

solar radiation may be predicted approximately on the assumption that the absorption coefficient is independent of ν. To attain this approximate constancy Woolley equates the absorption coefficients at wave-lengths 4000 A. and 6000 A. He further assumes that the opacity at the respective wave-lengths arises from absorption by metallic atoms and by hydrogen atoms in state 3. He finds, from this condition, that $f_c = 2{\cdot}37 \times 10^{-6}$. It is obvious that this low value, whether it be correct or not, arises directly from the high excitation potential of the 3-quantum state of hydrogen. So few hydrogen atoms exist in this state that the absorbing power of the metallic atoms, despite their relative scarcity, must be cut far below the value to be expected from Kramers' theory. The value is made even lower through omission of series limits from higher quantum states.

We have already presented arguments against the small values of f_c. It should be noted, however, that Woolley's assumptions lead to extremely low opacities and would require electron pressures at the base of the reversing layer at least 10^3 times greater than those observed. While we do not wish to place any special emphasis on the fact, it is interesting to note that our formula, (2.26), approximately fulfills Woolley's requirement of equal opacities at wave-lengths 4000 A. and 6000 A.

In his paper Woolley also discusses the question of continuity of the absorption coefficient. Near the series limit the overlapping lines give an absorption coefficient that goes over, without discontinuity, into the spectral region to the violet of the limit. Some of the values given in Woolley's Table III do not exhibit this continuity. Woolley does not question the existence of the continuity, but he lays the disagreement to possible defects in the wave-mechanical calculations. We find that Sugiura's formulæ for the individual sub-series of hydrogen, as well as our general formulæ for the integrated transitions, require *exact* continuity in all cases. Any disagreement must be attributed to numerical errors. Woolley's formula for testing continuity holds only when the ν^{-3} law is strictly obeyed. In the general case, the condition for continuity requires

$$\lim_{\kappa \to \infty} \left(\frac{df}{d\nu} \right) = \lim_{n \to \infty} \left(\frac{n^3}{2RZ^2} f_{nn'} \right). \tag{5.4}$$

Woolley's f_c should not be confused with our f values for the continuum. The reader should be warned that it will not generally be legitimate to treat the continuum as if it were a single discrete state with an f value equal to f_c. The appropriate weighting factors, ϖ_e, must be introduced to take account of the velocity distribution of the electrons and the effect of electron pressure.

Appendix

Asymptotic Expansions for

$$\Delta \equiv F(-n+1,\ -n',\ 1,\ x)^2 - F(-n'+1,\ -n,\ 1,\ x)^2,\ x = -4nn'/(n-n')^2$$

Asymptotic series for Δ for large values of n and n' can be obtained by the method of steepest descent. For this purpose it is necessary to represent the

hypergeometric functions by integrals in the complex plane. Starting with the second function appearing in Δ, we first write $F(-n'+1, -n, 1, x) = (1-x)^{n+n'}F(n', n+1, 1, x) \equiv (1-x)^{n+n'}P$, where P satisfies the differential equation

$$x(1-x)\ddot{P} + [1-x(2+n+n')]\dot{P} - n'(n+1)P = 0. \qquad \text{A (1)}$$

If we now assume that

$$P = (1/2\pi i)\int[(1-z)^n/z^{n'}(1-x-z)^{n+1}]dz, \qquad \text{A (2)}$$

and substitute in (1), we shall obtain the expression

$$\int \frac{d}{dz}\left[\frac{(1-z)^{1+n}}{z^{n'-1}(1-x-z)^{2+n}}\right]dz = 0.$$

Hence the above form for P will be correct provided we choose the path of integration such that the difference in the values of the quantity

$$[(1-z)^{1+n}/z^{n'-1}(1-x-z)^{2+n}]$$

at the beginning and the end of the path is zero. Similarly one finds that

$$F(-n+1, -n', 1, x) = (1-x)^{n+n'}(1/2\pi i)\int[(1-z)^{n-1}/(1-x-z)^n z^{n'+1}]dz \equiv$$
$$\equiv (1-x)^{n+n'}Q, \qquad \text{A (3)}$$

provided that the increment in $[(1-z)^n/z^{n'}(1-x-z)^{n+1}]$ over the path is zero.

We shall take up first the case when n and n' are positive integers. The F's in Δ then reduce to Jacobi polynomials,* for which we have

$$F(-n', -n+1, 1, x) = \frac{(1-x)^{n+n'}}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}[x^{n-1}(1-x)^{-n'-1}] =$$
$$= \frac{(1-x)^{n+n'}}{n'!}\frac{d^{n'}}{dx^{n'}}[x^{n'}(1-x)^{-n}] = \frac{(1-x)^{n+n'}}{2\pi i}\oint_{t=x}\frac{t^{n'}(1-t)^{-n}dt}{(t-x)^{n'+1}} =$$
$$= \frac{(1-x)^{n+n'}}{2\pi i}\oint_{z=1-x}\frac{(1-z)^{n'}dz}{z^n(1-x-z)^{n'+1}}, \qquad \text{A (4)}$$

where the path in the z plane is a negative (clockwise) loop enclosing $z = (1-x)$ but not z and 1. Since, with n and $n' \geqslant 1$, the integral in A (4) vanishes on a circle of infinite radius, the above loop can be deformed into a positive loop enclosing 1 and 0, but not $(1-x)$. This path, which evidently satisfies the aforementioned conditions, we shall adopt for P and Q.

Following the usual procedure of the method of steepest descent, we write :

$$Q \equiv \left(\frac{1}{2\pi i}\right)\oint\frac{(1-z)^{n-1}dz}{z^{n'+1}(1-x-z)^n} = \left(\frac{1}{2\pi i}\right)\oint\frac{e^{-n'\left[\ln z - \frac{1}{\alpha}\ln\left(\frac{1-z}{1-x-z}\right)\right]}dz}{z(1-z)},$$

$$P \equiv \left(\frac{1}{2\pi i}\right)\oint\frac{(1-z)^n dz}{z^{n'}(1-x-z)^{n+1}} = \left(\frac{1}{2\pi i}\right)\oint\frac{e^{-n'\left[\ln z - \frac{1}{\alpha}\ln\left(\frac{1-z}{1-x-z}\right)\right]}dz}{(1-x-z)}, \qquad \text{A (5)}$$

* Courant and Hilbert, *Meth. d. Math. Physik*, p. 74, 1924.

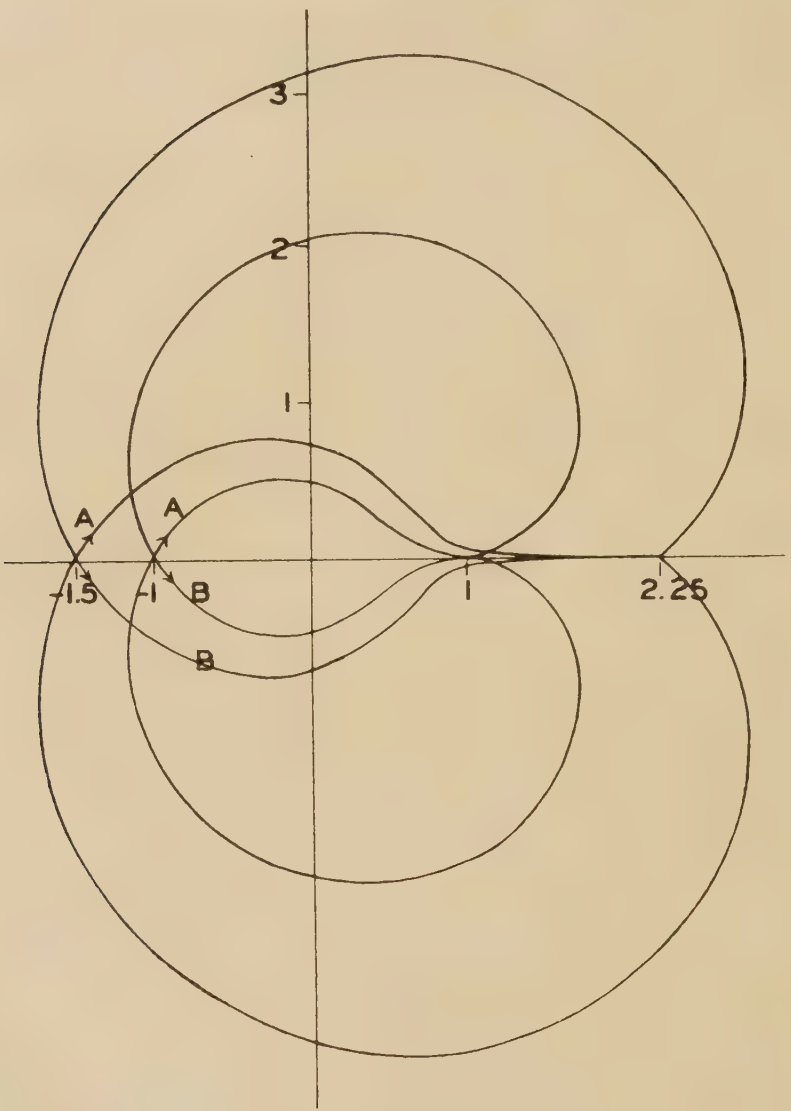

FIGURE 2

Paths of steepest descent and ascent in case n, n′ are real and $\alpha = 0, 1/5$.

with $\alpha \equiv (n'/n)$. The derivative with respect to z of the exponents in A (5) vanishes at $z = -(1+\alpha)/(1-\alpha) \equiv -\beta$. At this point

$$f(z) \equiv lnz - (1/\alpha)[ln(1-z) - ln(1-x-z)] = i\pi + (1+1/\alpha)ln\beta.$$

The path of steepest descent, which starts at $z = -\beta$, is such that along it the imaginary part of $f(z)$ equals $i\pi$. If we write $z = \rho e^{i\theta}$, this requires that

$$\theta - (1/\alpha)\tan^{-1}\{x\rho \sin\theta/[(1-x)(1-2\rho\cos\theta) - x\rho\cos\theta + \rho^2]\} = \pi, \qquad \text{A (6)}$$

or

$$\rho = \tfrac{1}{2}[(2-x)\cos\theta - x\sin\theta/\tan(\alpha\pi - \alpha\theta)] \pm$$
$$\pm \{(\tfrac{1}{4})[(2-x)\cos\theta - x\sin\theta/\tan(\alpha\pi-\alpha\theta)]^2 - 1 + x\}^{\frac{1}{2}}. \quad \text{A (7)}$$

A (7) determines two loops, symmetrical around the real axis, both starting at $-\beta$ and ending at $(1-x)$. These are shown in fig. 2 for the cases $\alpha = 0$, $\alpha = (1/5)$. To find the branches along which $f(z)$ increases we define $\tau(z)$, real along the path, by

$$\tau(z) \equiv lnz - (1/\alpha)[ln(1-z) - ln(1-x-z)] - i\pi - (1+1/\alpha)ln\beta, \quad \text{A (8)}$$

and develop τ in a power series of $u \equiv (z+\beta)$. One finds that

$$-\tau/(1-\alpha)^2 \equiv y^3 = (1/12\beta^3)u^3 + (1/8\beta^4)u^4 + (1/80\beta^5)(11+\alpha^2)u^5 +$$
$$+ (1/96\beta^6)(13+3\alpha^2)u^6 + \ldots \quad \text{A (9)}$$

When τ is positive,

$$y = |\tau/(1-\alpha^2)|^{1/3} e^{i\pi},\ e^{i\pi/3},\ e^{i5\pi/3}.$$

Now when $|u|$ is small $y \simeq (1/12\beta^3)^{1/3}u$, hence τ is positive along the direction indicated by arrows in fig. 2, where, in fact, it increases in a monotone fashion from zero to ∞. Clearly, a path which encloses 1 and 0 is represented by $B - A$. To avoid the singularity at $(1-x)$ one may deform the path in its vicinity to a vertical line, such a modification being immaterial to the result on account of the exponential factor in A (5). With τ as an independent variable, we now have

$$Q = (1/2\pi i)e^{-i\pi n'}\beta^{-(n'+n)}\int_0^\infty e^{-n'\tau}\left[(1-x-z)/(z+\beta)^2\right]_A^B d\tau, \quad \text{A (10)}$$

$$P = (1/2\pi i)e^{-i\pi n'}\beta^{-(n'+n)}\int_0^\infty e^{-n'\tau}\left[(z-z^2)/(z+\beta)^2\right]_A^B d\tau. \quad \text{A (11)}$$

The quantities in brackets in A (10) and A (11) can be obtained as functions of τ, or y, by first expanding in a power series in u and then expressing u in terms of y by inverting A (9). One thus obtains

$$\left[(1-x-z)/(z+\beta)^2\right]_A^B = \left\{\left(1+\frac{1}{\beta}\right)12^{-2/3}y^{-2} + 12^{-1/3}y^{-1} + \left[\left(1+\frac{1}{\beta}\right)\left(\frac{7}{20}+\frac{\alpha^2}{10}\right) - \frac{1}{2\beta}\right] + \right.$$
$$+ \frac{12^{1/3}(1+\alpha^2)}{20}y + \left(1+\frac{1}{\beta}\right)12^{2/3}\left(\frac{3}{560}+\frac{121}{700}\alpha^2+\frac{3\alpha^4}{560}\right)y^2 + \ldots \left.\right\}\Bigg|_{y=\frac{e^{i5\pi/3}\tau^{1/3}}{(1-\alpha)^2}}^{y=\frac{e^{i\pi/3}\tau^{1/3}}{(1-\alpha^2)^{1/3}}} =$$
$$= i\sqrt{3}\left\{\left(1+\frac{1}{\beta}\right)12^{-2/3}(1-\alpha^2)^{2/3}\tau^{-2/3} + 12^{-1/3}(1-\alpha^2)^{1/3}\tau^{-1/3} - \frac{12^{1/3}(1+\alpha^2)\tau^{1/3}}{20(1-\alpha^2)^{1/3}} - \right.$$
$$\left. - \left(1+\frac{1}{\beta}\right)12^{2/3}\left(\frac{3}{560}+\frac{121}{700}\alpha^2+\frac{3\alpha^4}{560}\right)\frac{\tau^{2/3}}{(1-\alpha)^{2/3}} + c\tau^{4/3}\right\}. \quad \text{A (12)}$$

Similarly one finds that

$$\left[\frac{z(1-z)}{(z+\beta)^2}\right]_A^B = i\sqrt{3}\left\{-\left(1+\frac{1}{\beta}\right)12^{-2/3}(1-\alpha^2)^{2/3}\tau^{-2/3} + 12^{-1/3}(1-\alpha^2)^{1/3}\tau^{-1/3} - \frac{12^{1/3}(1+\alpha^2)\tau^{1/3}}{20(1-\alpha^2)^{1/3}} + \right.$$
$$\left. + \left(1+\frac{1}{\beta}\right)12^{2/3}\left(\frac{3}{560}+\frac{121}{700}\alpha^2+\frac{3\alpha^4}{560}\right)\frac{\tau^{2/3}}{(1-\alpha^2)^{3/2}} + c\tau^{4/3} + \ldots\right\}. \quad \text{A (13)}$$

Upon carrying out the integration in A (10) and A (11) we now obtain

$$F(-n',\ -n+1,\ 1,\ x) = \frac{e^{-i\pi n'}\beta^{(n+n')}\sqrt{3}}{2\pi\,.\,12^{2/3}}\left\{\frac{\left(1+\frac{1}{\beta}\right)(1-\alpha^2)^{2/3}\Gamma(1/3)}{(n')^{1/3}} + \frac{12^{1/3}(1-\alpha^2)^{1/3}\Gamma(2/3)}{(n')^{2/3}} - \right.$$
$$\left. - \frac{3(1+\alpha^2)\Gamma(4/3)}{5(1-\alpha^2)^{1/3}(n')^{4/3}} - \frac{(1+1/\beta)12^{1/3}\Gamma(5/3)9}{(1-\alpha^2)^{2/3}(n')^{5/3}140}\left(1+\frac{484\alpha^2}{15}+\alpha^4\right) + \frac{c}{(n')^{7/3}}\right\}, \quad \text{A (14)}$$

$$F(-n'+1,\ -n,\ 1,\ x) = -\frac{e^{-i\pi n'}\beta^{(n+n')}\sqrt{3}}{2\pi\,.\,12^{2/3}}\left\{\frac{(1+1/\beta)(1-\alpha^2)^{2/3}\Gamma(1/3)}{(n')^{1/3}} - \frac{12^{1/3}(1-\alpha^2)^{1/3}\Gamma(2/3)}{(n')^{2/3}} + \right.$$
$$\left. + \frac{3(1+\alpha^2)\Gamma(4/3)}{5(1-\alpha^2)^{1/3}(n')^{4/3}} - \frac{(1+1/\beta)12^{1/3}\Gamma(5/3)9}{(1-\alpha^2)^{2/3}(n')^{5/3}140}\left(1+\frac{484\alpha^2}{15}+\alpha^4\right) - \frac{c}{(n')^{7/3}}\right\}. \quad \text{A (15)}$$

Finally we have

$$\Delta = F^2(-n',\ -n+1,\ 1,\ x)^2 - F^2(-n'+1,\ -n,\ 1,\ x) = \frac{\left(\frac{n+n'}{n-n'}\right)^{2n+2n'}(n-n')}{\sqrt{3}\pi nn'}\cdot$$
$$\cdot\left[1 - \frac{0{\cdot}1728(1+\alpha^2)}{(1-\alpha^2)^{2/3}(n')^{2/3}} - \frac{0{\cdot}0496\left(1+\frac{484\alpha^2}{15}+\alpha^4\right)}{(1-\alpha^2)^{4/3}(n')^{4/3}} + \ldots\right]. \quad \text{A (16)}$$

The degree of approximation of A (16) is shown in Table VI, where the last column gives the quantity in brackets. In the first case, the third term is of the order of magnitude of the second, so that a better approximation is obtained by stopping with the second term. Clearly the approximation is good when n' is large and α small.

TABLE VI

n'	n	α	Δ exact	Δ appr.	
10	20	$\frac{1}{2}$	$3{\cdot}6672 \times 10^{26}$	$3{\cdot}676 \times 10^{26}$	$1 - 0{\cdot}0564 - 0{\cdot}0286$
20	100	$\frac{1}{5}$	$1{\cdot}3087 \times 10^{40}$	$1{\cdot}3069 \times 10^{40}$	$1 - 0{\cdot}0251 - 0{\cdot}0022$
10	∞	0	$4{\cdot}1557 \times 10^{17}$	$4{\cdot}1548 \times 10^{17}$	$1 - 0{\cdot}03723 - 0{\cdot}00230$
20	∞	0	$4{\cdot}9654 \times 10^{32}$	$4{\cdot}96710 \times 10^{32}$	$1 - 0{\cdot}02346 - 0{\cdot}00091$

There now remains the discussion of the case of free-free transitions, where $n = ik$, $n' = il$, and $l > k$. Here too x is a negative real number. The

integrands in A (5), which now have branch points at 0, 1 and $1-x$, can be made single-valued by cutting the z plane along the real axis from 1 to $(1-x)$ and from 0 to $-\infty$ and assigning values for $arg(z)$ as shown in fig. 3. In this plane the expressions $[(1-z)^{1+n}/z^{n'-1}(1-x-z)^{2+n}]$ and $[(1-z)^n/z^{n'}(1-x-z)^{n+1}]$ return to their original values after passing a loop around the cut from 1 to $(1-x)$. The integrals A (2) and A (3), when taken along this path, are then solutions of the differential equations satisfied by P and Q respectively, and since, when we pass the loop in the negative direction, they assume the value $+1$ for $x=0$, they actually represent P and Q. Having chosen the appropriate path, we find that P and Q are still given by A (5), where now $n'=ik$ and $\alpha=(k/l)$. If we write

$$\psi(z) \equiv ilnz - i(1/\alpha)[ln(1-z) - ln(1-x-z)],$$

we find again that $\dot{\psi}(z)$ vanishes at $z_0=-\beta$ and that $\psi(z_0)=-\pi+i(1+1/\alpha)ln\beta$. The path of steepest descent is then chosen such that along it $Im.\,\psi(z)=(1+1/\alpha)ln\beta$. With $z=\rho e^{i\theta}$ this implies that

$$ln\rho+\left(\frac{1}{2\alpha}\right)ln\{[(1-x)^2-2\rho(1-x)\cos\theta+\rho^2]/(1-2\rho\cos\theta+\rho^2)\}=(1+1/\alpha)ln\beta,$$

or that

$$\cos\theta=\frac{(1-x)^2+\rho^2-u(1+\rho^2)}{2\rho(1-x-u)}, \qquad u\equiv\beta^{2+2\alpha}/\rho^{2\alpha}. \qquad \text{A (17)}$$

In the limit of $\alpha\to 0$ the path is given by

$$\cos\theta=[(1+\rho^2)(ln\rho-2)+4]/(2\rho ln\rho). \qquad \text{A (18)}$$

These consist of two loops, symmetrical about the axis of reals, starting at $z_0=-\beta$ (either on the upper or on the lower bank) and ending on the positive axis of reals at distances less than 1 and greater than $(1-x)$ respectively, as is shown for the case $\alpha=0$ in fig. 3. We now define $\tau(z)$, real along the path, by

$$\tau(z)=ilnz-i(1/\alpha)[ln(1-z)-ln(l-x-z)]-i(1+1/\alpha)ln\beta+\pi, \qquad \text{A (19)}$$

and develop $\tau(z)$ into a power series in $u=(z+\beta)$. Clearly, we have $y^3=i\tau/(1-\alpha^2)$, $y=\left|\frac{\tau}{1-\alpha^2}\right|^{1/3}e^{i\pi/6}$, $e^{i5\pi/6}$, $e^{i3\pi/2}$, with y^3 given by A (9).

Hence, starting with $z=\beta e^{i\pi}$, τ will be positive along A and B, along each of which it in fact increases from 0 to 2π, its final value at $z=\beta e^{-i\pi}$. The appropriate path for our integrals is $B-A$, since this forms a closed negative loop around the cut $1\to(1-x)$, and does not cross the cut to the left of the origin. We now have instead of A (10) and A (11),

$$Q=-\left(\frac{1}{2\pi}\right)e^{\pi\kappa}\beta^{-i(\kappa+l)}\int_0^{2\pi}e^{-\kappa\tau}\left[(1-x-z)/(z+\beta)^2\right]_A^B d\tau, \qquad \text{A (20)}$$

$$P=-\left(\frac{1}{2\pi}\right)e^{\pi\kappa}\beta^{-i(\kappa+l)}\int_0^{2\pi}e^{-\kappa\tau}\left[(z-z^2)/(z+\beta)^2\right]_A^B d\tau, \qquad \text{A (21)}$$

where at B $y=e^{i5\pi/6}\left|\frac{\tau}{(1-\alpha^2)}\right|^{1/3}$ and at A $y=e^{i\pi/6}\left|\frac{\tau}{(1-\alpha^2)}\right|^{1/3}$, but the

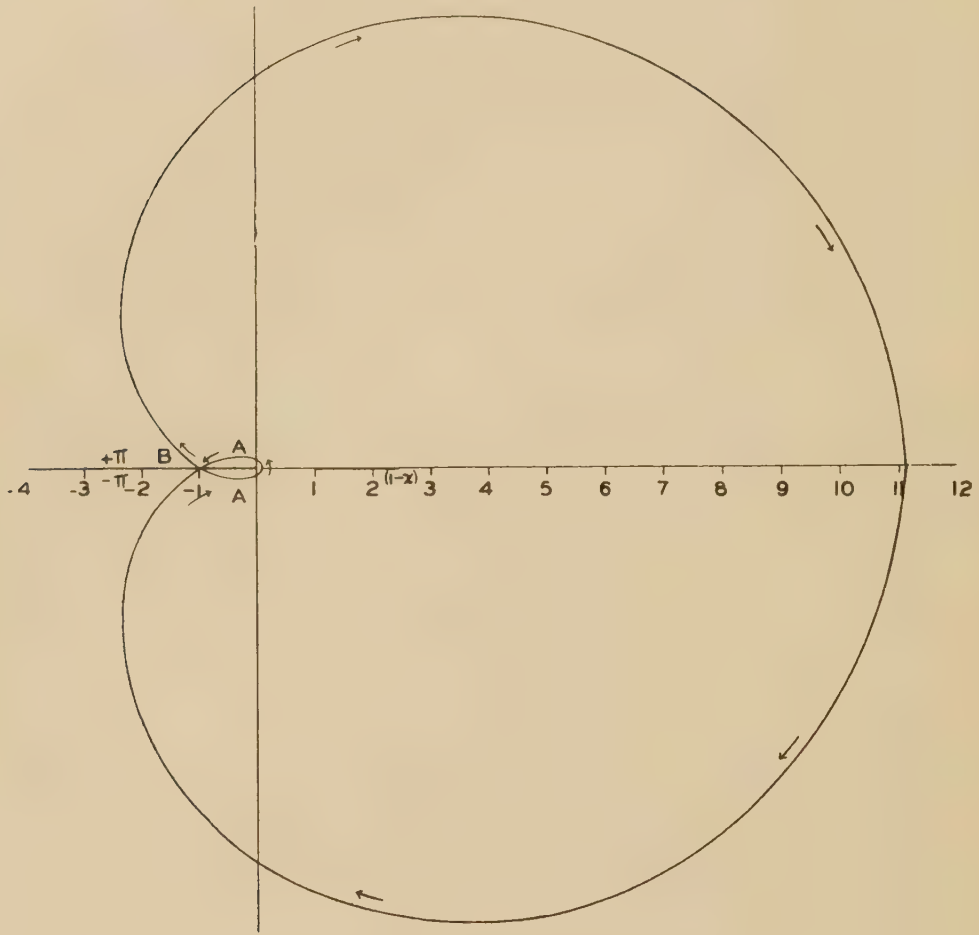

FIGURE 3

Path of steepest descent in case $n=i\kappa$, $n'=il$, and $\alpha=0$. The positive sense of loop A is clockwise. When $\alpha\neq 0$, $(1-x)$ will be to the right of 1, as indicated in the figure.

expressions in brackets in A (20) and A (21) are the *same* functions of y as before. In the subsequent integrations we neglect terms of the order of $e^{-2\pi k}$ by making the upper limits in the integral ∞, since for large values of κ they are of lower order of magnitude than any power of κ. We finally get

$$\Delta = \frac{ie^{2\pi\kappa}(l-\kappa)\left(\dfrac{l+\kappa}{l-\kappa}\right)^{2i(\kappa+l)}}{lk\pi\sqrt{3}}R,$$

$$R = \left[1 + \frac{0{\cdot}1728(1+\kappa^2/l^2)}{(1-\kappa^2/l^2)^{2/3}\kappa^{2/3}} - \frac{0{\cdot}0496\left(1+\dfrac{484\kappa^2}{15l^2}+\kappa^4/l^4\right)}{(1-\kappa^2/l^2)^{4/3}\kappa^{4/3}}\right], \qquad \text{A (22)}$$

$$|\Delta| = \frac{e^{2\pi\kappa}(l-\kappa)R}{\sqrt{3}\pi l\kappa}. \qquad \text{A (23)}$$

From A (22) we now get an expression for Δ for the case of a free-discrete transition, by substituting n' for il. The result is

$$|\Delta| = \frac{e^{4\kappa \tan^{-1}(n'/\kappa)}(\kappa^2+n'^2)^{1/2}}{\sqrt{3}\pi\kappa n'}\left[1 - \frac{0{\cdot}1728(1-n'^2/\kappa^2)}{(1+n'^2/\kappa^2)^{2/3}(n')^{2/3}} - \frac{0{\cdot}0496\left(1-\frac{484}{15}\frac{n'^2}{\kappa^2}+\frac{n'^4}{\kappa^4}\right)}{(1+n'^2/\kappa^2)^{4/3}(n')^{4/3}}\right], \qquad \text{A (24)}$$

a formula which is applicable for $\kappa \gtrless n'$, provided n' is large. When $\kappa < n'$, A (24) can be written in the convenient form

$$|\Delta| = \frac{e^{4\kappa \tan^{-1}(n'/\kappa)}(\kappa^2+n'^2)^{1/2}}{\sqrt{3}\pi\kappa n'}\left[1 + \frac{0{\cdot}1728(1-\kappa^2/n'^2)}{(1+\kappa^2/n'^2)^{2/3}\kappa^{2/3}} - \frac{0{\cdot}0496\left(1-\frac{484\kappa^2}{n'^2}+\frac{\kappa^4}{n'^4}\right)}{(1+\kappa^2/n'^2)^{4/3}\kappa^{4/3}}\right]. \qquad \text{A (25)}$$

It should be noticed that A (24) and A (25) can also be derived from A (16) by putting $n = i\kappa$; * also that A (16) and A (23) are inapplicable when $n \simeq n'$ and $\kappa \simeq l$, while this is not true for A (24) and A (25) when $\kappa \simeq n'$. This difference is to be expected on physical grounds, since the former two cases represent transitions between states of nearly equal energies, while the other does not.

The case $\kappa \simeq l$ (κ and l finite) deserves special consideration since it corresponds to the long-wave limit in continuous-continuous transitions. The usual procedure used in the method of steepest descent for this case fails because of the infinity in x: with $c \equiv l-\kappa$, $x = -4\kappa l/c^2$. On account of this very fact, however, we can use here the asymptotic expansions of the hypergeometric functions in powers of (l/x). We have †

$$\Delta = F(-n+1, -n', 1, x)^2 - F(-n'+1, -n, 1, x)^2 =$$
$$= \left\{\frac{\Gamma(-1+n-n')(4\kappa l/c^2)^{-1+n}}{\Gamma(n)\Gamma(-n')}F(1-n, 1-n, 2-n+n', 1/x) + \right.$$
$$\left. + \frac{\Gamma(1-n+n')(4\kappa l/c^2)^{n'}}{\Gamma(1-n)\Gamma(1+n')}F(-n', -n', -n'+n, 1/x)\right\}^2 -$$

---

* One can, in fact, obtain A (22) from A (16) by retaining in the latter the factor $e^{-i2\pi n'}$ and writing il for n and $i\kappa$ for n'. This purely formal substitution is merely suggested by the fact that $|n| > |n'|$ and $|l| > |\kappa|$, and has no physical significance. Further, one has to write $(n')^{2/3} = -(\kappa^{2/3})$ in the expression in the brackets. With these substitutions one gets the form A (22) for Δ, where we should keep in mind that with the present significance of n and n' the Δ in A (16) is the negative of the Δ in A (22).

† Whittaker and Watson, *Modern Analysis*, p. 289, 1920. This formula should read:

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a, b, c, z) = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)}(-z)^{-a}\,.\,F(a, 1-c+a, 1-b+a, 1/z) +$$
$$+\,.\,\frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)}(-z)^{-b}F(b, 1-c+b, 1-a+b, 1/z).$$

$$-\left\{\frac{\Gamma(-1+n'-n)(4\kappa l/c^2)^{-1+n'}}{\Gamma(n')\Gamma(-n)}F(1-n',\ 1-n',\ 2-n'+n,\ 1/x)+\right.$$
$$\left.+\frac{\Gamma(1-n'+n)(4\kappa l/c^2)^n}{\Gamma(1-n')\Gamma(1+n)}F(-n,\ -n,\ -n+n',\ 1/x)\right\}^2 \simeq$$
$$\simeq\left[\frac{\Gamma(1+ic)(4\kappa l/c^2)^{n'}}{\Gamma(1-n)\Gamma(1+n')}\right]^2-\left[\frac{\Gamma(1-ic)(4\kappa l/c^2)^n}{\Gamma(1-n')\Gamma(1+n)}\right]^2.$$

Further, since

$$\Gamma(1+ic)\simeq 1,\qquad \Gamma(1-n)\Gamma(1+n')\simeq \kappa l\Gamma(-i\kappa)\Gamma(i\kappa)\simeq\frac{2\pi\kappa}{e^{\pi\kappa}(1-e^{-2\pi\kappa})},$$

we get

$$\Delta=\frac{(4\kappa l/c^2)^{2n}}{\kappa^4\Gamma^2(-i\kappa)\Gamma^2(i\kappa)}[e^{i2c\ln(4\kappa l/c^2)}-1]=icl n(2\kappa/c)(4\kappa l/c^2)^{2i\kappa}e^{2\pi\kappa}(1-e^{-2\pi\kappa})^2/(\pi^2\kappa),$$

$$|\Delta|=\frac{cl n(2\kappa/c)e^{2\pi\kappa}(1-e^{-2\pi\kappa})^2}{\pi^2\kappa^2}. \qquad \text{A (26)}$$

Finally we shall discuss the case where n and n' are integers and $c(\equiv n-n'\geqslant 1)$ is relatively small. The expansion A (16) is then inapplicable since α is close to unity; in fact, the expression in brackets in A (16) for (g_I) deviates here considerably from unity. We use again the hypergeometric functions of $(1/x)$, which in this case can be obtained by writing the terminating series in the reverse order and factoring out the term with the highest power of x. Thus

$$F(-n',\ -n+1,\ 1,\ x)=\frac{(n-1)!\,x^{n'}}{n'!\,(c-1)!}F(-n',-n',\ c,\ 1/x), \qquad \text{A (27)}$$

$$F(-n'+1,\ -n,\ 1,\ x)=\frac{n!\,x^{n'-1}}{(n'-1)!\,(c+1)!}F(-n'+1,-n'+1,\ c+2,\ 1/x). \qquad \text{A (28)}$$

When c is small, the F's on the right-hand sides of A (27) and A (28) converge rapidly, and one can quickly compute the individual terms appearing in Δ. In this connection we may note that

$$(1-x)^{2n'}=x^{2n'}(1-1/x)^{2n'}=x^{2n'}(1+c^2/2n\,.\,2n')^{2n'}\simeq x^{2n'}e^{\frac{c^2}{2n}}. \qquad \text{A (29)}$$

Abstract

1. A general formula is given for $f_{nn'}$, the oscillator strength of a transition between atomic states n and n'. When the upper state lies in the continuum, n is set equal to $i\kappa$, where κ is real, and the absolute value of the resulting complex expression is taken. Asymptotic expressions for various limiting cases are worked out by the method of steepest descents (see Appendix). Gaunt's limiting formula is verified and extended. The expression

$$f_{nn'}=\frac{2^6}{3\sqrt{3}\pi}\,\frac{1}{\varpi_{n'}}\,\frac{1}{\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)}\left|\frac{1}{n^3}\cdot\frac{1}{n'^3}\right|g,$$

where $\varpi_{n'}$ is the weight of the lower state and g a correction factor of the order unity, holds for any atomic transition. Tables and formulæ for g are given for all important cases. Tables of $f_{nn'}$ and $A_{nn'}$ (Einstein transition probability) are given for the first fourteen series of H. Those for the Lyman, Balmer and Paschen series are particularly complete. A new formulation is given of the general theory of statistical equilibrium and the photoelectric effect.

2. Various expressions for the continuous absorption coefficient arising from free-free and bound-free transitions are derived. These are summed over all quantum states, extended to cover the case of complex atoms, and the resulting formulæ are applied to calculation of the opacity of the solar atmosphere. The electron pressure so derived is in good agreement with observation.

3. The Rosseland mean opacity coefficient is calculated and extended to include the case of a gas that is capable of being multiply ionized.

4. The opacity that arises from interaction of neutral hydrogen atoms and free electrons is evaluated and is shown to be negligible except, perhaps, in the atmospheres of very cool stars.

5. Woolley's conclusion, that the opacity arising from ionization of metallic atoms is 10^{-6} times less than that to be expected on Kramers' theory, is shown to be false.

Harvard College Observatory and
Massachusetts Institute of Technology :
April 1935.

STELLAR PARALLAXES DETERMINED AT THE ROYAL OBSERVATORY, GREENWICH.

(Communicated by the Astronomer Royal)

The following list is a continuation of the results obtained with the 26-inch Thompson refractor, and published annually in the November number of the *Monthly Notices*.

The visual magnitude and spectral type for stars contained in the *Draper Catalogue* have been extracted from that catalogue. For the fainter stars, they have been taken from other sources. The photographic magnitudes are from the *Greenwich Astrographic Catalogue*.

Two values of the proper motion in R.A. are given. The first (as also the proper motion in Dec.) is absolute, and is based on meridian observations or the comparison of long-interval photographs. The second is the relative proper motion derived from the parallax plates.

In the last column the first value of the probable error refers to the parallax, and the second to the proper motion in R.A. These are based on the value of the probable error for a plate of unit weight given in the eighth column. The mean probable error for unit weight is $\pm 0''.025$.

To minimise the effect of quasi-seasonal error the observations have been spread over not less than six epochs. The average number of plates per star is 28; for those stars observed during the summer months the number is in excess of the average to compensate for the loss of factor.

Generally, six comparison stars of mean magnitude 11–12 have been used, the magnitude of the parallax star being reduced to equality by means of a rotating sector. For the brighter stars a neutral-tinted filter absorbing 4·8 magnitudes is used in conjunction with the sector. Two exposures of 3 minutes each are made on each plate, the carrier being turned through 180° after the first exposure. As the filter is permanently mounted on one of the carriers the reversal should eliminate any distortion due to the filter as well as to film displacement.

No.	Name and B.D. No.	Mag. Vis. Photog.	Type	R.A. Dec. (1900)	Proper Motion R.A.	Proper Motion Dec.	P.E. Unit Weight Unit	Parallax Unit	P.E. Unit
					Unit o″·001		o″·001	o″·001	o″·001
				h m s					
1	70° 35	10·3		0 34 17	+337	+ 20	21	+ 6	± 6
		9·8		+70° 29′	+328				± 2
2	79° 38	10·3		1 12 5	+282	− 87	25	+38	± 7
		10·6		+79° 38′	+281				± 6
3	64° 416	10·0		3 49 34	+134	−249	25	− 5	± 8
		9·7		+64° 49′	+107				± 7
4	73° 210	6·7	Ko	3 55 21	+ 21	− 80	24	+12	± 6
		7·2		+73° 44′	+ 50				± 7
5	Gr. 763	7·2	Ko	3 57 9	+ 54	− 4	17	+ 7	± 4
	78° 146	7·8		+78° 46′	+ 56				± 5
6	Gr. 779	5·6	Ko	4 9 38	+ 26	− 20	20	+10	± 5
	80° 133	6·4		+80° 35′	+ 4				± 5
7	C. 618	7·4	Ko	4 21 32	0	− 42	23	+ 7	± 6
	83° 114	7·9		+83° 50′	+ 15				± 6
8	O.A. 5813	8·0	G5	5 20 48	+ 1	− 50	25	+ 4	± 7
	70° 355	9·0		+70° 44′	− 2				± 6
9	O.A. 6702	9·2	Ko	6 13 52	− 8	− 60	29	+ 5	± 7
	70° 399	9·6		+70° 6′	− 2				± 5
10	A.C. 72° 3338	...		6 32 56	−111	−544	26	+29	± 6
		12·4		+72° 0′	−145				± 4
11	Gr. 1365	7·7	Ko	7 45 54	+ 7	− 58	26	+ 3	± 7
	78° 273	8·5		+77° 55′	+ 5				± 6
12	P. VII 187	5·3	Ao	7 49 4	− 23	− 81	30	+ 9	± 7
	79° 265	5·5		+79° 45′	− 26				± 7

No.	Name and B.D. No.	Mag. Vis. Photog.	Type	R.A. Dec. (1900)	Proper Motion R.A.	Proper Motion Dec.	P.E. Unit Weight Unit	Parallax Unit	P.E. Unit
				h m s	Unit 0″·001		0″·001	0″·001	0″·001
13	Gr. 1400	6·6	F8	7 59 58	+142	+121	22	+43	± 6
	70° 497	6·9		+70° 1′	+133				± 6
14	Gr. 1414	7·0	F2	8 6 19	− 9	− 72	27	+ 8	± 7
	66° 541	7·0		+66° 29′	− 8				± 5
15	Gr. 1418	7·4	F2	8 25 20	−105	− 98	27	+14	± 7
	85° 128	7·6		+85° 24′	−103				± 6
16	75° 343	10·3		8 25 32	−113	−220	24	+ 7	± 6
		9·7		+74° 54′	− 97				± 5
17	L.F. 1312	7·4	F8	8 35 21	−138	−128	29	+22	± 8
	84° 186	7·8		+84° 16′	−149				± 8
18	Br. 1731	5·8	A0	12 48 16	− 15	+ 38	30	−12	± 9
	84° 289	6·2		+83° 58′	+ 10				± 5
19	Br. 1732	5·3	A2	12 48 23	− 25	+ 36	31	+ 4	± 9
	84° 290	5·7		+83° 57′	+ 11				± 5
20	P. XII 255	6·0	F5	12 57 53	−180	+ 19	33	+42	± 11
	64° 927	6·3		+64° 9′	−148				± 5
21	L.F. 2161	7·7	K0	13 13 35	− 48	+ 9	21	−13	± 7
	79° 418	8·3		+79° 14′	− 11				± 4
22	A.C. 66° 4310	...		14 20 18	−205	+245	24	+23	± 8
		10·6		+66° 1′	−203				± 3
23	O.A. 15135	8·3	G5	15 4 5	−114	+ 98	28	+49	± 9
	64° 1046	8·7		+64° 26′	−115				± 4
24	L.F. 2871	7·9	F5	17 4 25	− 19	+ 74	24	0	± 8
	69° 891	8·0		+69° 56′	− 24				± 3
25	C. 2624	8·8	K0	17 25 59	+ 2	+ 13	21	− 8	± 7
	82° 518	8·9		+82° 26′	− 13				± 5
26	L. 32554	8·4	K0	17 39 35	− 19	+120	25	+38	± 10
	65° 1203	9·0		+65° 3′	− 15				± 8
27	C. 2727	10·3		17 57 29	+ 1	− 7	22	+13	± 9
	85° 290	11·4		+85° 19′	− 3				± 6
28	Br. 2295	6·8	K0	18 6 10	− 30	+ 21	25	+34	± 10
	64° 1245	7·4		+64° 12′	− 38				± 3
29	L. 34166	7·4	K0	18 19 47	+ 24	+ 17	20	−16	± 8
	64° 1263	7·9		+64° 9′	+ 24				± 4

No.	Name and B.D. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion R.A.	Proper-Motion Dec.	P.E. Unit Weight Unit	Parallax	P.E.
				h m s	Unit 0″·001		0″·001	Unit 0″·001	Unit 0″·001
30	L.F. 3222	8·3	Ko	19 30 49	+ 65	+ 92	27	+ 4	± 9
	68° 1071	8·2		+68° 32′	+ 50				± 4
31	L. 38195	8·5	Ko	19 47 42	− 11	+ 93	22	+16	± 8
	77° 743	9·1		+77° 17′	+ 5				± 4
32	Gr. 3085	6·9	Ko	20 5 51	+ 21	+ 20	31	+19	± 11
	73° 897	7·6		+73° 37′	+ 27				± 6
33	L.F. 3509	6·8	Ko	20 30 55	+ 22	+ 3	26	0	± 8
	64° 1449	7·3		+65° 3′	+ 23				± 4
34	L. 40593	9·1	Ko	20 48 6	+ 20	+ 64	27	−15	± 9
	75° 756	9·3		+75° 37′	+ 39				± 5
35	C. 3207	9·5	G5	20 52 5	+ 31	+ 36	24	−19	± 7
	85° 355	9·1		+85° 28′	+ 22				± 4
36	71° 1053	9·8		21 10 40	+214	+248	31	+12	± 10
		9·8		+71° 14′	+223				± 6
37	P. XXI 88	6·8	Ko	21 12 42	+ 7	− 18	25	+ 7	± 7
	69° 1152	7·2		+69° 37′	+ 1				± 3
38	Gr. 3477	7·3	F8	21 21 32	+168	+126	29	+ 4	± 9
	79° 701	7·5		+79° 55′	+178				± 4
39	Gr. 3826	6·0	F2	22 30 9	+108	+ 56	24	− 8	± 8
	69° 1262	6·0	A5	+69° 24′	+107				± 5
40	A.C. 72° 9714	...		23 19 50	+310	+ 92	23	+18	± 7
		12·3		+72° 15′	+295				± 5
41	A.C. 67° 9282	...		23 30 25	+359	− 4	32	+ 8	± 11
		11·3		+67° 45′	+356				± 8

Nos. 18, 19. Σ 1694.
No. 38. Σ 5801 not separable.

STELLAR PARALLAXES DETERMINED PHOTOGRAPHICALLY AT THE CAPE OBSERVATORY. (SEVENTH LIST.)

(Communicated by H.M. Astronomer)

For references to the previous lists see *M.N.*, 95, 70, 1934.

The form of publication is the same as that of previous lists.

The Durchmusterung number given is B.D. down to $-23°$, Co.D. south of this limit to $-52°$, and C.P.D. further south.

The proper-motion in R.A. given in the second line for each star is derived from the photographs; the proper-motion in R.A. and Dec. in the first line is from meridian observations or other sources. The two values of probable error given in the last column refer to the parallax and photographic proper-motion in R.A. respectively.

Two of the parallaxes exceed $0''.100$, of which one is a first determination. For seven stars the parallax is negative, including that of two stars which were measured merely because they were near other stars for which the parallax was being determined. One large proper-motion has not been confirmed.

The parallaxes are relative to the mean of four or five comparison stars which are generally between magnitude 11·0 and 12·0.

No.	Name and D.M. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion R.A.	Proper-Motion Dec.	P.E. Unit Weight. Unit	Parallax. Unit	P.E. Unit
				h m s	Unit $0''.001$		$0''.001$	$0''.001$	$0''.001$
601	$-27°$ 223	7·73	Go	0 39 29	+ 343	+135	29	+37	± 11
		8·3		$-27°$ 4′·0	+ 302				± 13
602	Boss 285	5·21	Fo	1 9 22	+ 125	+268	29	+78	± 11
	$-8°$ 216	5·49		$-8°$ 27′·6	+ 99				± 12
603	$-52°$ 242	6·07	F8	1 53 11	+ 340	+250	24	+38	± 8
		6·7		$-52°$ 15′·7	+ 359				± 9
604	U.O.C. 48	...	...	2 36 35	+ 262	+210	26	+ 6	± 9
		11		$-58°$ 1′·4	+ 156				± 10
605	Boss 631	4·61	F5	2 40 26	+ 323	+ 47	27	+44	± 9
	$-19°$ 518	5·03		$-18°$ 59′·7	+ 311				± 10
606	Ci 20° 184	...	...	2 42 29	+ 451	+210	24	+ 9	± 7
		10		$-58°$ 47′·4	+ 428				± 8
607	$-63°$ 231	8·0	Ko	3 25 18	+ 344	−250	18	+82	± 7
		9·0		$-63°$ 50′·3	+ 386				± 7

No.	Name and D.M. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion R.A.	Proper-Motion Dec.	P.E. Unit Weight. Unit	Parallax. Unit	P.E. Unit
					Unit o″·001		o″·001	o″·001	o″·001
608	Anon.	...	...	h m s 5 0 45	...	...	27	−44	± 10
		12·5		−65° 54′	− 22				± 10
609	U.O.C. 53	...	...	5 0 57	+ 141	+314	24	+11	± 9
		12·5		−65° 55′	+ 125				± 9
610	Boss 1459	3·22	Ko	5 47 26	+ 47	+394	22	+47	± 7
	−35° 2546	4·22		−35° 48′·4	+ 76				± 8
611	Boss 1609	1·99	B1	6 18 18	− 7	0	24	+22	± 7
	−17° 1467	1·77		−17° 54′·4	− 16				± 9
612	−32° 3010	8·9	...	6 21 7	+ 102	+410	26	+41	± 9
		...		−32° 4′·7	+ 135				± 11
613	−1° 1265	9·4	...	6 26 21	− 315	−390	21	+10	± 8
		...		−1° 29′·4	− 233				± 8
614	−5° 1844	6·77	K2	6 47 26	− 552	− 20	22	+136	± 10
		7·84		−5° 3′·2	− 545				± 10
615	−0° 1520	9·6	Go	6 53 30	+ 360	−630	19	+41	± 8
		10·2		−0° 19′·7	+ 352				± 9
616	−56° 1199	7·4	F8	6 54 5	− 49	+600	25	+17	± 9
		8·1		−56° 49′·6	− 8				± 10
617	Boss 1827	5·80	Go	7 0 53	− 120	+370	29	+75	± 10
	−43° 2906			−43° 28′·1	− 96				± 11
618	Boss 1828	6·92	Go	7 0 55	− 116	+362	24	+56	± 9
	−43° 2907			−43° 28′·3	− 91				± 10
619	Anon.	...	...	8 7 19	...	...	34	−19	± 11
		12		−52° 40′·3	+ 3				± 13
620	U.O.C. 48	...	...	8 7 28	− 573	+589	26	+56	± 8
		12		−52° 41′·2	− 507				± 9
621	Boss 2303	5·20	Ao	8 33 36	− 13	− 12	27	+12	± 8
	−25° 6356	5·20		−25° 54′·3	+ 4				± 9
622	−12° 2918	9·5	...	9 26 26	+ 759	+ 70	30	+77	± 11
		...		−13° 3′·0	+ 773				± 11
623	−57° 2699	9·4	...	10 4 37	− 286	−250	27	+15	± 10
		...		−57° 59′·5	− 195				± 10
624	−0° 2326	8·9	Ko	10 15 42	− 660	−160	26	− 2	± 10
		9·9		−0° 57′·7	− 625				± 10

No.	Name and D.M. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion R.A.	Proper-Motion Dec.	P.E. Unit Weight.	Parallax.	P.E.
				h m s	Unit ″·001		Unit	Unit ″·001	Unit ″·001
625	Boss 3013	5·34	F5	11 21 23	− 311	− 79	24	+41	± 8
	−63° 1893	5·2		−63° 25′·2	− 289				± 9
626	−68° 1684	7·3	G5	12 27 49	− 501	−200	25	+87	± 9
		8·1		−68° 11′·7	− 539				± 10
627	−11° 3361	6·89	G5	12 41 20	− 299	+ 36	26	+ 9	± 8
		7·67		−11° 16′·0	− 253				± 10
628	−9° 3595	7·7	G5	12 53 54	− 829	+192	22	+37	± 9
		8·5		−9° 18′·0	− 783				± 10
629	Boss 3425	6·66	G5	13 7 14	− 205	−314	26	+ 5	± 9
	−31° 10156	8·0		−31° 20′·0	− 223				± 10
630	−57° 6166	8·4	B9	13 31 25	− 119	+270	26	+ 3	± 8
		8·1		−57° 41′·8	− 10				± 8
631	−45° 8719	10·0	...	13 42 48	− 317	−100	23	− 9	± 9
		...		−45° 14′·1	− 303				± 10
632	Boss 4137	5·56	G0	16 10 11	+ 222	−500	33	+58	± 11
	−7° 4242	6·12		−8° 6′·3	+ 180				± 13
633	−54° 7765	6·77	K2	16 28 54	+ 320	0	29	− 7	± 8
		8·4		−54° 46′·7	+ 3				± 10
634	−3° 3968	9·8	G5	16 29 27	− 120	−660	25	+ 46	± 8
		10·6		−4° 0′·1	− 135				± 10
635	Boss 4272	2·36	K0	16 43 41	− 616	−258	28	+ 51	± 8
	−34° 11285	3·36		−34° 6′·7	− 620				± 10
636	−27° 11772	9·5	...	17 32 11	− 494	−230	30	+ 10	± 8
		...		−27° 8′·0	− 476				± 11
637	−3° 4233	9·2	K5	17 59 48	+ 431	−312	22	+129	± 6
		10·4		−3° 2′	+ 551				± 8
638	Wolf 1463	...	...	18 28 5	− 149	−630	27	+ 39	± 9
	Ci 20° 1092	12		−6° 57′·5	− 109				± 11
639	−40° 12743	9·7	...	18 36 45	− 80	−510	27	+ 5	± 8
		...		−40° 9′·2	− 74				± 9
640	−23° 14813	9·2	G5	18 48 11	− 276	−610	24	− 2	± 7
		8·8		−23° 1′·1	− 143				± 9
641	−37° 12969	7·9	G5	18 48 30	+ 131	−380	29	+ 23	± 9
		8·1		−37° 36′·9	+ 128				± 11

No.	Name and B.D. No.	Mag. Vis. Phot.	Type	R.A. Dec. (1900)	Proper-Motion R.A.	Proper-Motion Dec.	P.E. Unit Weight Unit	Parallax Unit	P.E. Unit
				h m s	Unit 0″·001		0″·001	0″·001	0″·001
642	−0° 3676	9·3	Go	19 8 14	− 315	−430	30	+ 4	± 8
		9·9		−0° 45′·1	− 302				± 11
643	U.O.C. 48	...	...	19 29 28	0	−307	19	+ 9	± 5
		12		−58° 13′·8	+ 88				± 7
644	−38° 13832	6·92	Go	19 57 24	+ 35	−390	24	+ 27	± 7
		7·0		−38° 8′·7	+ 101				± 8
645	−14° 6448	5·27	G5	23 13 51	+ 268	− 94	20	+ 53	± 7
	(pr. comp.)	6·05		−13° 59′·9	+ 292				± 6
646	−19° 6489	8·03	Go	23 35 24	+ 304	− 54	23	+ 24	± 10
		8·4		−19° 32′·5	+ 337				± 6
647	−8° 6177	9·7	Go	23 38 24	+ 623	−166	30	− 7	± 11
		10·3		−8° 28′·7	+ 586				± 8
648	−48° 14610	7·04	Go	23 42 7	− 395	−220	22	+ 21	± 8
		8·0		−48° 49′·5	− 381				± 5
649	−17° 6856	8·5	G5	23 54 12	+1144	0	22	+ 18	± 8
		9·3		−17° 30′·0	+1023				± 8
650	−50° 14095	9·5	Go	23 54 30	− 29	−420	23	0	± 9
		9·6		−50° 23′·3	− 37				± 6

CHANGE FROM 5″ TO 3″ IN THE REDUCTION OF OCCULTATIONS OBSERVED IN 1936.

Ernest W. Brown and Dirk Brouwer.

Beginning with the first lunation of 1936 (New Moon, 1935 December 25) it is requested that 3″ be added to the Moon's mean longitude instead of the 5″ used during the past three years. This is equivalent to the addition of $0{\cdot}00152\,\Delta\alpha$ and $0{\cdot}00152\,\Delta\delta$, where $\Delta\alpha$ and $\Delta\delta$ are the variations per hour of the Moon's position (the variation per minute being no longer printed), or of $0^{h}{\cdot}00152 = 0^{m}{\cdot}091 = 5^{s}{\cdot}5$ to the time at which the Moon's position is obtained from the ephemeris.

The *Nautical Almanac* and the *American Ephemeris* for 1936 give for the first time an additional list of occultation stars between magnitudes 6·6 and 7·5. The positions of these stars were obtained from a catalogue of zodiacal stars prepared by J. C. Hammond from observations made during the years 1927–1930 with the 6-inch transit circle of the Naval Observatory at Washington. For the brighter occultation stars the positions published in the *Nautical Almanac* are obtained from this same catalogue. It is suggested to computers of reductions of occultations observed during 1936 that the positions of the brighter stars as given in the *Nautical Almanac* be used, in preference to positions obtained from Hedrick's Zodiacal Catalogue.

It is also suggested that in future publications of occultation reductions the angle χ be given as well as $\chi - \rho$, in order to facilitate computations of the effects of altered star positions.